

Notes on β -deformations of the pure spinor superstring in $AdS_5 \times S^5$

Oscar A. Bedoya[†], L. Ibiapina Beviláqua[†], Andrei Mikhailov[‡]
and Victor O. Rivelles[†]

[†] *Instituto de Física, Universidade de São Paulo,
C.P. 66.318 CEP 05315-970, São Paulo, SP, Brasil*
 \mathcal{E}

[‡] *Institute for the Physics and Mathematics of the Universe
University of Tokyo, Kashiwa, Chiba 277-8582, Japan*
 \mathcal{E}

[†] *Institute for Theoretical and Experimental Physics,
117259, Bol. Cheremushkinskaya, 25, Moscow, Russia*

Abstract

We study the properties of the vertex operator for the β -deformation of the superstring in $AdS_5 \times S^5$ in the pure spinor formalism. We discuss the action of supersymmetry on the infinitesimal β -deformation, the application of the homological perturbation theory, and the relation between the worldsheet description and the spacetime supergravity description.

Contents

1	Introduction	3
1.1	Deformations of the pure spinor action	4
1.2	Special case of β -deformation	5
1.2.1	First order in ε	5
1.2.2	Second order in ε	6
1.3	Types of β -deformations	7
1.3.1	Real β -deformations	7
1.3.2	Obstructed β -deformations	7
1.3.3	Complex β -deformations	8
1.4	Plan of the paper	9
1.5	Open questions	10
2	Notations and various identities	10
2.1	Notations	10
2.2	Various identities	11
2.3	BRST transformation of S_0	13
2.4	Adding antifields	15
3	Vertex corresponding to β-deformation	16
3.1	The vertex and its descent	16
3.2	What happens to the integrated vertex when B^{ab} is proportional to $f^{ab}{}_c A^c$? .	18
4	Constraint on the internal commutator	19
4.1	Additional constraint on B	19
4.2	Example of an unintegrated vertex violating the constraint	20
4.3	What happens in the flat space limit	21
5	Symmetries of the vertex	22
5.1	Representation theory	23
5.1.1	Matrix notations	23
5.1.2	Exterior product of two adjoint representations	23
5.1.3	Complex structure	24
5.2	Our vertex is not covariant	25
6	General deformation theory	25
6.1	Some general notations	25
6.2	Deforming with integrated vertex operator	26
6.2.1	BRST invariance at the first order in ε	26
6.2.2	BRST invariance at the second order in ε	26
6.2.3	The descent of $Q_1 V_1^{(2)}$	27
6.2.4	Going back	27

6.3	Comment about Q_2	28
6.4	Higher orders of perturbation theory	28
6.4.1	Going forward (necessary condition)	28
6.4.2	Going back (sufficient condition)	29
7	Applying the general theory to beta-deformation	31
7.1	Calculation of Q_1	31
7.2	Deformation of the BRST current	34
7.2.1	General procedure for calculating the current density	34
7.2.2	Particular case of BRST transformation	34
7.2.3	Conservation of the deformed current	35
7.3	Relation between $W_2^{(0)}$ and the Schouten bracket on $\Lambda^\bullet \mathfrak{g}$	36
7.4	Could a nonzero $[[B, B]]$ be harmless?	37
7.4.1	Operator $W_2^{(0)}$ may be Q -exact	37
7.4.2	But $W^{(0)}$ is of ghost number 3; isn't it always Q_0 -exact?	38
7.4.3	Open problem: how to tell if $W_2^{(0)}$ is BRST-exact in the covariant subcomplex?	39
7.5	Calculation of $V_2^{(2)}$	40
7.5.1	The case when $[[B, B]] = 0$	41
7.5.2	The case when $[[B, B]]$ is of the form (210)	43
7.6	Taking into account nonzero classical values of the antifields	43
7.6.1	Second order correction to the deformed action	43
7.6.2	BRST transformation of the shifted antifields	44
7.7	Conclusion	44
7.8	Comments	45
7.8.1	About higher orders	45
7.8.2	About the gauge transformation $B^{ab} \mapsto B^{ab} + f^{ab}_c G^c$	45
8	Properties of the Schouten bracket on $\Lambda^\bullet \mathfrak{g}$	45
8.1	Projection to $\mathfrak{g} \otimes \mathfrak{g}$	45
8.2	From the r -matrix point of view	46
8.3	The space of solutions to $[[B, B]] = 0$	47
8.3.1	The solutions of Maldacena-Lunin type	47
8.3.2	Solutions of more general type	47
8.4	Calculation of the bracket for elements of $6_{\mathbb{C}}^{u(3)} \subset 45_{\mathbb{C}}^{su(4)}$	48
9	Complex β-deformation	48
9.1	Complex structure in $45_{\mathbb{C}}$	48
9.2	Comparison with the quadratic obstruction found by Aharony, Kol, and Yankielowicz [1]	49
9.2.1	Formula for obstruction suggested in [1]	49
9.2.2	Comparison with $[[B, B]]$	49

9.2.3	Supersymmetric extension of (282)	50
10	Reading the supergravity fields from the vertex	51
10.1	The action of Berkovits and Howe	52
10.2	NS-NS B -field	55
10.3	RR B -field	56
10.3.1	How we measure the RR field strength	56
10.3.2	Steps 1 and 2: calculate the deformation of $P_{\alpha\hat{\beta}}$	57
10.3.3	Digression: integrating in d and \hat{d}	58
10.3.4	Step 3: examine the torsion components $T_{\alpha\beta}^m$ and $\hat{T}_{\hat{\alpha}\hat{\beta}}^m$ and do the necessary field redefinitions of w and λ	59
10.3.5	Conclusion: RR field strength is $*$ -dual in TS^5 to NSNS B -field	60
10.3.6	Digression: the coupling of $w\lambda$ to the NSNS 3-form	60
11	Relation to the description by Alday–Arutyunov–Frolov	64
12	General relation between NSNS and RR fields in the β-deformed solution	65
12.1	At the linearized level	65
12.2	Exact relation for the full solution	66
A	Antisymmetric tensor product of two adjoint representations of $su(4)$	67
A.1	As a representation of $su(4)$	67
A.2	As a representation of $u(3) \subset su(4)$	69
B	BRST operator in the near flat space expansion	69
C	Proof of a technical Lemma used in Section 7.4.3	70

1 Introduction

Historically, the development of the pure spinor formalism was mostly concentrated on the special case of flat space. But in fact the flat space case is a degenerate case. In many ways the general background is qualitatively different, the flat space being a special degenerate limit. The general, “typical” background has a non-degenerate Ramond-Ramond bispinor field. Among such non-degenerate examples the most symmetric one is $AdS_5 \times S^5$. Therefore the study of this background is important for the string theory in general.

During the last several years, continuous progress has been made in this direction. One of the observations made recently in [2] is that the pure spinor Lagrangian is invariant under the action of the global symmetry group $PSU(2, 2|4)$. This is in contrast with the case of flat space, where the Lagrangian is invariant only up to total derivatives. This observation was generalized in [3] where it was argued that the vertex operators for massless supergravity states can be chosen in a $PSU(2, 2|4)$ -covariant way.

At this time there are two explicit examples of vertices: the vertex for the zero mode of the dilaton (the descent of the Lagrangian) introduced in [4] and the vertex for the β -deformation introduced in [3]. In this paper we will study the vertex for the β -deformation. We will be mostly concerned with the following subjects:

- how the supersymmetry acts on β -deformations
- extension of an infinitesimal β -deformation to a finite β -deformation; the homological perturbation theory
- the space-time picture

First steps towards the pure spinor description of the β -deformed $AdS_5 \times S^5$ were made in [5], although our approach is somewhat different¹.

We will now briefly outline our paper.

1.1 Deformations of the pure spinor action

The Type IIB string worldsheet theory, in the pure spinor formulation, has the following structure:

1. An action S which is assumed to be local and conformally invariant;
2. A pair of BRST operators Q_L and Q_R with the properties:

$$Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0.$$

The “total” BRST operator Q is the sum of Q_L and Q_R :

$$Q = Q_L + Q_R ;$$

3. Two ghost number operators gh_L and gh_R , such that $gh_L(Q_L) = 1$, $gh_L(Q_R) = 0$, $gh_R(Q_L) = 0$, and $gh_R(Q_R) = 1$;
4. The composite b -ghost b_{++} , b_{--} , which satisfy:

$$\{Q, b_{++}\} = T_{++} \quad , \quad \{Q, b_{--}\} = T_{--}.$$

¹The authors of [5] followed the method of twisted boundary conditions previously used in [6, 7, 8] in the context of Green-Schwarz approach. We are using a more straightforward approach, using the vertex operator and the homological perturbation theory.

Given a worldsheet theory with these axioms satisfied, we ask ourselves: how can such a theory be deformed? It turns out that the infinitesimal deformations are parametrized by *integrated vertex operators*² $V_1^{(2)}$:

$$\begin{aligned} S &= S_0 + \varepsilon \int V_1^{(2)} \\ Q &= Q_0 + \varepsilon Q_1, \end{aligned} \tag{1}$$

where S_0 is the undeformed original action, invariant under Q_0 .

The integrated vertex operator should be a total derivative under the original BRST transformation:

$$Q_0 V_1^{(2)} \simeq d(\text{smth}). \tag{2}$$

where \simeq means that “equals on-shell”. The condition (2) guarantees that the deformed action is BRST-invariant at the first order; notice that the BRST transformation itself gets deformed, unless (2) is satisfied off-shell (which is usually not the case).

Generally speaking, given the first infinitesimal deformation $V_1^{(2)}$, it should be possible to construct the series:

$$\begin{aligned} S_{exact} &= S_0 + \varepsilon \int V_1^{(2)} + \varepsilon^2 \int V_2^{(2)} + \dots \\ Q_{exact} &= Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots \end{aligned} \tag{3}$$

and obtain the full deformed theory.

1.2 Special case of β -deformation

In this paper we will consider an example: the so-called β -*deformations*. These deformations were introduced in field theories by Leigh and Strassler in [9].

1.2.1 First order in ε

Let us first consider the β -deformation at the linearized level. In the pure spinor formalism the corresponding vertex operator has a very simple form [3]:

$$V_1^{(2)} = \frac{1}{2} B^{ab} j_{[a} \wedge j_{b]}, \tag{4}$$

where j_a are the conserved currents corresponding to the global symmetries, and B^{ab} is a constant antisymmetric tensor, the parameter of the deformation³:

$$B \in (\mathfrak{g} \wedge \mathfrak{g})_0 / \mathfrak{g}, \tag{5}$$

²The subindex 1 in $V_1^{(2)}$ show that this is the 1-st infinitesimal deformation, and the superindex (2) indicates a 2-form.

³Here we consider the full supermultiplet of the linearized β -deformations. To the best of our knowledge, the orbits of the β -deformations under the supersymmetry have not been previously studied. But there is a construction of the deformations of the AdS part of $AdS_5 \times S^5$ in [10, 11], which must be related to the deformations of the sphere by the supersymmetry.

where:

- $\mathfrak{g} = psu(2, 2|4)$ is the global symmetry algebra; indices a, b enumerate the generators of \mathfrak{g}
- the subindex 0 means that the “inner commutator” is zero, see Eq. (115)
- the subspace $\mathfrak{g} \subset (\mathfrak{g} \wedge \mathfrak{g})_0$ is generated by $f_a^{bc} t_b \wedge t_c$ (for $B \in \mathfrak{g} \subset \mathfrak{g} \wedge \mathfrak{g}$ we find that (4) is a total derivative); in other words we consider B_1 and B_2 equivalent if:

$$B_1^{ab} - B_2^{ab} = f^{ab}{}_c G^c \quad (6)$$

We want to construct the series of the form (3) so that Q_{exact} is a symmetry of S_{exact} and $Q_{exact}^2 = 0$. It follows from the general principles of string theory, that this should be always possible starting from the first order $V_1^{(2)}$ given by (4).

1.2.2 Second order in ε

The second order correction $V_2^{(2)}$ depends on B quadratically. It turns out that the dependence of $V_2^{(2)}$ on B is rather subtle. Notice that the space of linearized β -deformations (5) is fibered by the orbits of $PSU(2, 2|4)$. The structure of $V_2^{(2)}$ depends on which orbit B belongs to. The formula for $V_2^{(2)}$ is relatively simple when B satisfies a certain quadratic equation. This equation says that the Schouten-Nijenhuis-Gerstenhaber bracket $\llbracket B, B \rrbracket$ is equivalent to zero. The standard definition of this bracket is:

$$\llbracket B, B \rrbracket \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \quad (7)$$

$$\llbracket B, B \rrbracket^{abc} = B^{e[a} f^b{}_{ef} B^{c]f} \quad (8)$$

However this definition does not respect the equivalence relation (6). The construction which does respect this equivalence relation is this one:

$$\llbracket B, B \rrbracket \bmod L_\Delta \quad (9)$$

$$\text{where } L_\Delta \text{ is generated by } f^{[ab}{}_m A^{m|c]} \quad (10)$$

In our terminology, the β -deformation is called *real* if:

$$\llbracket B, B \rrbracket \in L_\Delta \quad (11)$$

We distinguish β -deformations of the following three types: *real*, *complex*, and *obstructed*.

1.3 Types of β -deformations

1.3.1 Real β -deformations

In the special case when $B \in \Lambda^2 \mathfrak{su}(4)$, the condition (11) for real β -deformations is equivalent to:

$$[[B, B]] = 0. \quad (12)$$

We explicitly constructed $V_2^{(2)}$ for real β -deformations in Section 7.5 Eqs. (237), (247). We find that $V_2^{(2)}$ is a polynomial function of the currents j and the group element g . For such B , we *conjecture* that the polynomial dependence of $V_n^{(2)}$ on the currents and the group element will persist at higher orders. This agrees with the formula for the obstruction suggested in [1]. In fact, we suspect⁴ that $V_2^{(2)}$ is always a polynomial function, but we point out that the formula is much simpler when (11) is satisfied. In fact we do not even know the explicit formula in the case when B does not satisfy (11).

Notice that for the real Maldacena-Lunin [12] solutions B satisfies a stronger condition:

$$B^{ea} f^b_{ef} B^{cf} = 0 \quad (13)$$

(no antisymmetrization of abc).

Does (12) imply (13)? The condition (12) can be interpreted as a classical Yang-Baxter equation for the r -matrix $r = B$ [13]. If this condition is satisfied, then the antisymmetric tensor B defines a left-invariant Poisson structure on the supergroup $PSU(2, 2|4)$. In this context the solutions of (12) have been previously studied in the mathematical literature. For a compact Lie group (such as $SU(4)$) it was proven in [14] that (12) implies that B lies in the exterior product of an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$:

$$B \subset \mathfrak{a} \wedge \mathfrak{a}. \quad (14)$$

This means that in this case (12) implies (13). However, for non-compact groups (such as $SU(2, 2)$) there are more general solutions. Solutions of the rank 8 for $SU(2, 2)$ were constructed in [15].

Therefore the results of [15] suggest that there are solutions more general than those considered in [12], corresponding to the deformation of the AdS part of $AdS_5 \times S^5$. But at this time we have not proven that such solutions would not be obstructed at the cubic and higher orders.

1.3.2 Obstructed β -deformations

What happens for a general B ?

Generally speaking, *any* solution of the linearized supergravity can be “repaired” to the full exact nonlinear solution, if we dress it appropriately with the corrections to self-interaction. In other words, it is always possible to construct the series of the form (3) order

⁴because it was proven in [1] that there the obstruction to the existence of a polynomial solution only appears at the third order in ϵ , thus we expect to have problems only with $V_n^{(2)}$ for $n > 2$

by order in ε . But for a generic B the nonlinear solution will not be a polynomial in the current and the group element. In particular, the solution for a general B will not be periodic in the global time of AdS_5 . In other words, the nonlinear solution will not be a universal cover of anything (while AdS_5 was a universal cover of the hyperboloid).

1.3.3 Complex β -deformations

It is natural to ask the following question: what is the condition on B necessary and sufficient for the nonlinear solution to be, order by order in ε , of the polynomial type?

At this time, we do not have a full answer to this question in our approach.

The condition (11) is probably sufficient, although we have only proven this at the order ε^2 . But it is not necessary. It appears too strong. For some B violating (11) there are still polynomial solutions. This can be seen using the solution-generating technique of [12]. As we will review in Sections 5.1.3 and 9.1 the space of linearized β -deformations has a complex structure, *i.e.* there is an operator \mathcal{I} commuting with the $PSU(2, 2|4)$ such that $\mathcal{I}^2 = -1$. The results of [12] imply that if B corresponds to a polynomial solution then $e^{\mathcal{I}\phi}B$ also corresponds to a polynomial solution. But the action of \mathcal{I} violates the condition (11). In our terminology, the complex β -deformations are those which can be connected to the real β -deformations by $e^{\mathcal{I}\phi}$.

The analysis of [1] implies that the obstruction first appears at the third order of perturbation theory (*i.e.* ε^3). The authors of [1] suggest the formula for the obstruction, which we review and supersymmetrize in Section 9.2 — see Eq. (282) and its supersymmetric generalization (288). It is interesting that this obstruction can *almost* be expressed in terms of the Schouten bracket, but not quite — see Section 9.2.3. It appears to us that there are cases when B satisfies (288) but is not in the orbit $e^{\mathcal{I}\phi}B$ of the real B . This means, provided that B satisfying (288) are indeed unobstructed, that not all of these solutions can be obtained by the solution-generating trick of [12] from the real solutions.

Complex β -deformations receive α' -corrections The known results from the field theory side [16] combined with the AdS/CFT correspondence imply that the complex solutions receive accumulating⁵ α' -corrections [6]. This suggests that there should be a proof of finiteness to all orders in α' which works for B satisfying (11) and does not work for the complex β -deformation.

The picture presented in the current literature [1, 12, 6, 16] (as we understand it) is the following:

- real β -deformations are periodic in global time, including the α' corrections
- complex β -deformations are classically periodic, but receive accumulating α' -corrections quantum mechanically

⁵these α' corrections are “accumulating” in the sense that the corrected background is not periodic in the global time of AdS; the deviation from periodicity corresponds to the anomalous dimension on the field theory side

- the β -deformations which we call “obstructed” are not periodic even classically

Brief review of the literature As pointed out in [12], one can generate the Lunin-Maldacena background by performing a TsT chain of transformation on the $AdS_5 \times S^5$ background: a T-duality transformation along one angular variable φ_1 , a shift in another angular variable φ_2 and again a T-duality along φ_1 . The parameter of the deformation is introduced by the shift and it is therefore real. On the other hand, the β -deformed field theory is allowed to have complex β [9]. In order to generate a complex parameter in the dual geometry, we have to apply S-duality before and after the TsT chain, so we would have a STsTS chain (for further discussion, see [17]).

Note that, since we perform twice the S-duality transformation, the original and the deformed solutions are in the same coupling regime. However, as it was pointed out by Frolov in [7], the S-duality step departs from the world-sheet treatment, as opposed to T-duality. Indeed, as it is discussed in Frolov, Roiban and Tseytlin’s paper [6], the T-duality can be implemented directly at the level of the world-sheet, so the starting point may be the classical Green-Schwarz action on $AdS_5 \times S^5$. They also say that there are no good reason to believe that the S-dual background will not be deformed by α'/R^2 corrections, while TsT does not introduce any correction. Indeed, while T-duality and a coordinate shift preserve the 2d conformal invariance of the string theory, with S-duality things are very different and we may need to modify the classical superstring action by extra α'/R^2 correction terms in order to ensure its quantum 2d conformal invariance.

1.4 Plan of the paper

In Section 2 we list basic formulas for the pure spinor superstring in $AdS_5 \times S^5$ and briefly discuss the descent procedure. Then in Section 3 we introduce the vertex operator which corresponds to the β -deformation at the linearized level. In Section 4 we discuss an additional constraint on the parameter B , which we don’t understand as well as we would want to. We then discuss the symmetries of the vertex in Section 5, with a surprising conclusion that our vertex is not strictly speaking covariant. In Section 6 we discuss the general deformation theory of the classical worldsheet action, and in Section 7 apply it to the β -deformation. In particular, in Sections 7.5 and 7.6 we obtain the explicit formula for the second order correction $V_2^{(2)}$. It turns out that the Schouten bracket $\llbracket B, B \rrbracket$ plays an important role, and we further study its properties in Section 8. In Section 9 we discuss the complex β -deformations and the equation for the obstruction proposed by Aharony, Kol and Yankielowicz in [1]; we discuss the supersymmetric generalization of their formula. In Section 10 we show (at the linearized level) that the target space supergravity fields of our worldsheet theory agree with the known supergravity description of the β -deformation. In Section 11 we explain (at the linearized level) the relation between our approach and the approach of [6, 7, 8, 5] which uses the twisted boundary conditions. In Section 12 we discuss an interesting general relation between the RR fields and the NSNS fields for the β -deformed background.

1.5 Open questions

We will list here several open questions:

1. The constraint on the internal commutator described in Section 4 has to be explained.
2. Even under the condition (11) we have only constructed the action up to the second order in ϵ . It should be possible to find an explicit expression for $V_n^{(2)}$ for $n > 2$.
3. Is it true that the condition for the existence of the classical periodic solution is given by Eq. (282) of Section 9.2? We must understand the relation to the covariant subcomplex of [3] suggested in Section 7.4.2.
4. It would be nice to prove the nonrenormalization theorem for real β -deformations without invoking the TsT -transformations argument of [17]. In the language of twisted boundary conditions, which we review in Section 11, how do we derive (11)? It has to be related to the BRST symmetry of the twisted boundary conditions.
5. Solutions of [15] mentioned in Section 1.3.1 have to be studied explicitly. Are they obstructed at the higher orders of ϵ ?

2 Notations and various identities

In this Section we will use [18, 19, 20].

2.1 Notations

The global symmetry algebra is $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$. It has the \mathbf{Z}_4 grading $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$. Various worldsheet fields take values in \mathfrak{g} ; the lower index will denote the \mathbf{Z}_4 grade of the field. For example, consider this field:

$$w_{1+}$$

The index 1 means that it takes values in \mathfrak{g}_1 , and the index $+$ means that it has the conformal dimension $(1, 0)$; similarly the field w_{3-} has the conformal dimension $(0, 1)$ and takes values in \mathfrak{g}_3 .

The pure spinor ghosts are λ_3 and $\tilde{\lambda}_1$. The tilde over $\tilde{\lambda}_1$ is redundant; it is to stress that this field would be right-moving in the free field limit. We will then sometimes write λ and $\tilde{\lambda}$ for short.

The corresponding conjugate momenta are w_{1+} and \tilde{w}_{3-} ; the kinetic term in the action is given by Eq. (304) below. Once again, our notations are highly excessive because there is no such things as for example w_{1-} . Therefore, we could have just written w_+ and w_- instead of w_{1+} and \tilde{w}_{3-} .

The pure spinor action is constructed out of the right-invariant current

$$J = -dgg^{-1}, \tag{15}$$

which is invariant under $g \rightarrow gH$, with $H \in PSU(2, 2|4)$ being a global parameter.

We use notations from Section 2 of [20]. As in that paper, the spectral parameter of the Lax operator will be denoted z . The Lax equation is

$$[\partial_+ + J_+[z], \partial_- + J_-[z]] = 0, \quad (16)$$

where

$$J_+[z] = J_{0+} - N_+ + z^{-1}J_{3+} + z^{-2}J_{2+} + z^{-3}J_{1+} + z^{-4}N_+ \quad (17)$$

$$J_-[z] = J_{0-} - N_- + zJ_{1-} + z^2J_{2-} + z^3J_{3-} + z^4N_-. \quad (18)$$

The Lax connection $J_\pm[z]$ defined above should not be confused with the current $J_\pm = -\partial_\pm g g^{-1}$ on the right-hand side. The current does not depend on the Lax parameter, while the Lax connection does.

We will also introduce l by

$$l = \log z. \quad (19)$$

Then the density of the global conserved charges can be written as

$$j = g^{-1} \frac{dJ}{dl} \Big|_{l=0} g. \quad (20)$$

Following Berkovits and Howe, we also denote z, \bar{z} the worldsheet coordinates. We think that the meaning of z will be always clear from the context.

For any $x \in \mathfrak{g}$ we denote:

$$x_a = \text{Str}(x t_a) \quad (21)$$

In particular:

$$j_a = \text{Str} \left(g^{-1} \frac{dJ}{dl} \Big|_{l=0} g t_a \right) \quad (22)$$

2.2 Various identities

In calculations involving supersymmetry, it is very convenient (and in fact suggested by the definitions) to introduce a set of sufficiently many formal anticommuting constant parameters:

$$\epsilon, \epsilon', \epsilon'', \dots \quad (23)$$

For example, the superalgebra $\mathfrak{gl}(m|n; \mathbf{C})$ consists of the block matrices $\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$ where $a_{00}, a_{10}, a_{01}, a_{11}$ are $m \times m, m \times n, n \times m$ and $n \times n$ matrices, respectively, and moreover a_{00} and a_{11} are constructed from polynomials over \mathbf{C} involving even number of ϵ -parameters (23) and a_{10} and a_{01} involve odd number of ϵ . The commutator is then the usual commutator $[x, y] = xy - yx$. We will use such parameters in our notations; they should not be confused with the small parameter in (3) which is denoted ε .

The BRST charge⁶ acts on the group element g as

$$\epsilon Q g = (\epsilon \lambda_3 + \epsilon \tilde{\lambda}_1) g \quad (24)$$

and, because of (15) and (17)-(18),

$$\epsilon Q J = -D(\epsilon \lambda + \epsilon \tilde{\lambda}) \quad (25)$$

$$\epsilon Q z \partial_z J = D(\epsilon \lambda - \epsilon \tilde{\lambda}) - [z \partial_z J, \epsilon \lambda + \epsilon \tilde{\lambda}], \quad (26)$$

where

$$D = \partial + [J, \quad]. \quad (27)$$

It will be useful to define the “composite” ghosts

$$\Lambda(\epsilon) = g^{-1}(\epsilon \lambda - \epsilon \tilde{\lambda})g, \quad \bar{\Lambda}(\epsilon) = g^{-1}(\epsilon \lambda + \epsilon \tilde{\lambda})g. \quad (28)$$

Remind that $Q = Q_L + Q_R$, and denote $\bar{Q} = Q_L - Q_R$. We then have

$$\epsilon Q g = g \bar{\Lambda}(\epsilon), \quad \epsilon \bar{Q} g = g \Lambda(\epsilon), \quad (29)$$

$$\epsilon Q g^{-1} = -\bar{\Lambda}(\epsilon) g^{-1}, \quad \epsilon \bar{Q} g^{-1} = -\Lambda(\epsilon) g^{-1}. \quad (30)$$

Because of the pure spinor constraint, $\epsilon \lambda \epsilon' \lambda = \epsilon \tilde{\lambda} \epsilon' \tilde{\lambda} = 0$, and thus

$$\Lambda(\epsilon) \Lambda(\epsilon') = -\bar{\Lambda}(\epsilon) \bar{\Lambda}(\epsilon'), \quad \Lambda(\epsilon) \bar{\Lambda}(\epsilon') = -\bar{\Lambda}(\epsilon) \Lambda(\epsilon') \quad (31)$$

(note also that the above expressions are antisymmetric under $\epsilon \leftrightarrow \epsilon'$).

Using the identities above, we show that

$$\begin{aligned} \epsilon Q \bar{\Lambda}(\epsilon') &= (\epsilon Q g^{-1})(\epsilon' \lambda + \epsilon' \tilde{\lambda})g + g^{-1}(\epsilon' \lambda + \epsilon' \tilde{\lambda})(\epsilon Q g) = \\ &= -\bar{\Lambda}(\epsilon) \bar{\Lambda}(\epsilon') + \bar{\Lambda}(\epsilon') \bar{\Lambda}(\epsilon) = \\ &= -[\bar{\Lambda}(\epsilon), \bar{\Lambda}(\epsilon')] = \\ &= [\Lambda(\epsilon), \Lambda(\epsilon')] = -\epsilon \bar{Q} \Lambda(\epsilon'), \end{aligned} \quad (32)$$

$$\epsilon Q \Lambda(\epsilon') = -[\bar{\Lambda}(\epsilon), \Lambda(\epsilon')] = 0, \quad (33)$$

$$\epsilon Q \Lambda(\epsilon') = -[\Lambda(\epsilon), \bar{\Lambda}(\epsilon')] = 0. \quad (34)$$

We may also write the above equations in $psu(2, 2|4)$ components:

$$\Lambda_a = \text{Str}(\Lambda t_a) \quad (35)$$

$$\epsilon \bar{Q} \Lambda_a(\epsilon') = -f_a^{bc} \Lambda_b(\epsilon) \Lambda_c(\epsilon'), \quad Q \Lambda_a = 0, \quad \epsilon Q \bar{\Lambda}_a(\epsilon') = -f_a^{bc} \bar{\Lambda}_b(\epsilon) \bar{\Lambda}_c(\epsilon'), \quad \bar{Q} \bar{\Lambda}_a = 0. \quad (36)$$

⁶In this section we will consider BRST transformations in the undeformed theory on $AdS_5 \times S^5$; we will omit the index 0 and write Q instead of Q_0 , to simplify the notations. All the formulas in this section are about the undeformed theory.

There are constraints on Λ and $\bar{\Lambda}$:

$$f_a^{bc} \Lambda_b \bar{\Lambda}_c = 0, \quad f_a^{bc} (\Lambda_b \Lambda_c + \bar{\Lambda}_b \bar{\Lambda}_c) = 0. \quad (37)$$

The “composite” ghosts satisfy also

$$Q (d\bar{\Lambda}(\epsilon') - 2[g^{-1}z\partial_z Jg, \Lambda(\epsilon')]) = 0, \quad (38)$$

since

$$\begin{aligned} d[\Lambda(\epsilon), \Lambda(\epsilon')] &= d[g^{-1}(\epsilon\lambda - \epsilon\tilde{\lambda})g, g^{-1}(\epsilon'\lambda - \epsilon'\tilde{\lambda})g] = \\ &= 2[g^{-1}D(\epsilon\lambda - \epsilon\tilde{\lambda})g, g^{-1}(\epsilon'\lambda - \epsilon'\tilde{\lambda})g] = \\ &= 2\left[g^{-1}\epsilon Q(z\partial_z J)g + g^{-1}[z\partial_z J, \epsilon\lambda + \epsilon\tilde{\lambda}g], g^{-1}(\epsilon'\lambda - \epsilon'\tilde{\lambda})g\right] = \\ &= 2\epsilon Q[g^{-1}z\partial_z Jg, g^{-1}(\epsilon'\lambda - \epsilon'\tilde{\lambda})g] \\ &= 2\epsilon Q[g^{-1}z\partial_z Jg, \Lambda(\epsilon')]. \end{aligned} \quad (39)$$

Finally, since the conserved charge j satisfies

$$j = g^{-1}z\partial_z Jg, \quad (40)$$

then,

$$\epsilon Qj = d(g^{-1}\epsilon\bar{Q}g), \quad (41)$$

$$\epsilon\bar{Q}j = d(g^{-1}\epsilon Qg); \quad (42)$$

and note that Eq. (38) can be written as

$$\Omega = \epsilon\bar{Q}j - 2[j, g^{-1}\epsilon\bar{Q}g] \in \text{Ker}(Q). \quad (43)$$

2.3 BRST transformation of S_0

$$\begin{aligned} S_0 = & \frac{R^2}{\pi} \int d^2z \text{Str} \left(\frac{1}{2} J_{2+} J_{2-} + \frac{3}{4} J_{1+} J_{3-} + \frac{1}{4} J_{3+} J_{1-} \right. \\ & \left. + w_{1+} \partial_- \lambda_3 + w_{3-} \partial_+ \lambda_1 + N_{0+} J_{0-} + N_{0-} J_{0+} - N_{0+} N_{0-} - w_{1+}^* w_{3-}^* \right), \end{aligned} \quad (44)$$

the ghost currents are:

$$N_{0+} = -\{w_{1+}, \lambda_3\}, \quad N_{0-} = -\{w_{3-}, \lambda_1\}, \quad (45)$$

The BRST transformations of the currents are:

$$\begin{aligned} \epsilon Q_L J_3 &= -D_0 \epsilon \lambda_3 \\ \epsilon Q_L J_2 &= -[J_3, \epsilon \lambda_3] \\ \epsilon Q_L J_1 &= -[J_2, \epsilon \lambda_3] \\ \epsilon Q_L J_0 &= -[J_1, \epsilon \lambda_3] \end{aligned} \quad (46)$$

Therefore the BRST transformation of the first row is:

$$\begin{aligned}
\text{Str} \quad & -\frac{1}{2}[J_{3+}, \epsilon\lambda_3]J_{2-} - \frac{1}{2}J_{2+}[J_{3-}, \epsilon\lambda_3] \\
& -\frac{3}{4}[J_{2+}, \epsilon\lambda_3]J_{3-} - \frac{3}{4}J_{1+}(D_{0-}\epsilon\lambda_3) \\
& -\frac{1}{4}(D_{0+}\epsilon\lambda_3)J_{1-} - \frac{1}{4}J_{3+}[J_{2-}, \epsilon\lambda_3]
\end{aligned} \tag{47}$$

We transform this as follows:

$$\begin{aligned}
\text{Str} \quad & -\frac{1}{4}D_{0+}(\epsilon\lambda_3 J_{1-}) + \frac{1}{4}D_{0-}(\epsilon\lambda_3 J_{1+}) + \frac{1}{4}\epsilon\lambda_3(D_{0+}J_{1-} - D_{0-}J_{1+}) \\
& -J_{1+}(D_{0-}\epsilon\lambda_3) \\
& -\frac{1}{2}[J_{3+}, \epsilon\lambda_3]J_{2-} - \frac{1}{2}J_{2+}[J_{3-}, \epsilon\lambda_3] \\
& -\frac{3}{4}[J_{2+}, \epsilon\lambda_3]J_{3-} - \frac{1}{4}J_{3+}[J_{2-}, \epsilon\lambda_3]
\end{aligned} \tag{48}$$

Let us use the Maurer-Cartan equation:

$$D_{0+}J_{1-} - D_{0-}J_{1+} + [J_{3+}, J_{2-}] + [J_{2+}, J_{3-}] = 0 \tag{49}$$

We get:

$$\begin{aligned}
\text{Str} \quad & -\frac{1}{4}\partial_+(\epsilon\lambda_3 J_{1-}) + \frac{1}{4}\partial_-(\epsilon\lambda_3 J_{1+}) \\
& -J_{1+}(D_{0-}\epsilon\lambda_3)
\end{aligned} \tag{50}$$

The term $-J_{1+}D_{0-}\epsilon\lambda_3$ cancels with the variation of $w_{1+}D_{0-}\lambda_3$ in the second row. We conclude:

$$\epsilon Q_L \mathcal{L} = -\frac{1}{4}d \text{Str}(\epsilon\lambda_3 J_1) \tag{51}$$

$$\epsilon Q \mathcal{L} = -\frac{1}{4}d \text{Str}(\epsilon\lambda_3 J_1 - \epsilon\lambda_1 J_3) \tag{52}$$

This equation is the first step of the descent procedure for the Lagrangian itself. The second step is:

$$\begin{aligned}
& \epsilon' Q \text{Str}(\epsilon\lambda_3 J_1 - \epsilon\lambda_1 J_3) = \\
& = \text{Str}(\epsilon\lambda_3 D_0 \epsilon' \lambda_1 - \epsilon\lambda_1 D_0 \epsilon' \lambda_3) = \\
& = \epsilon \epsilon' d \text{Str}(\lambda_3 \lambda_1)
\end{aligned} \tag{53}$$

Notice that Eq. (52) can be rewritten in the following way:

$$\epsilon Q \mathcal{L} = -\frac{1}{4}d \text{Str}(\epsilon \Lambda g^{-1} dg) \tag{54}$$

Using the notations of Section 6.1:

$$I_{\epsilon Q_0}^{(1)} = -\frac{1}{4}\text{Str}(\epsilon \Lambda g^{-1} dg) \tag{55}$$

2.4 Adding antifields

The Q defined so far is only nilpotent on-shell. Indeed, we get:

$$Qw_{1+} = -J_{1+} \quad , \quad QJ_{1+} = -D_{0+}\tilde{\lambda}_1 - [J_{2+}, \lambda_3] \quad (56)$$

To make Q nilpotent off-shell we have to introduce, following [21], the fermionic antifields w_{1+}^* and w_{3-}^* satisfying the constraints:

$$\{\tilde{\lambda}_1, w_{1+}^*\} = \{\lambda_3, \tilde{w}_{3-}^*\} = 0 \quad (57)$$

and modify the BRST transformations:

$$\begin{aligned} Qw_{1+} &= -J_{1+} - w_{1+}^* & Q\tilde{w}_{3-} &= -J_{3-} - \tilde{w}_{3-}^* \\ Qw_{1+}^* &= D_{0+}\tilde{\lambda}_1 - [N_+, \tilde{\lambda}_1] & Q\tilde{w}_{3-}^* &= D_{0-}\lambda_3 - [N_-, \lambda_3] \end{aligned} \quad (58)$$

With this modification, we obtain:

$$Q^2w_{1+} = [(J_{2+} + \{w_{1+}, \tilde{\lambda}_1\}), \lambda_3] + [\{\lambda_3, \tilde{\lambda}_1\}, w_{1+}] \quad (59)$$

which is a combination of the Lorentz gauge transformation and the pure-spinor-constraint gauge transformation of w .

Now we would like to modify the currents to include antifields. We propose:

$$\hat{j}_+ = g^{-1} \left(\frac{dJ_+}{dl} \Big|_{l=0} - 4w_{1+}^* \right) g \quad (60)$$

$$\hat{j}_- = g^{-1} \left(\frac{dJ_-}{dl} \Big|_{l=0} + 4w_{3-}^* \right) g \quad (61)$$

We need the off-shell version of Eq. (25), and with Q modified according to (58):

$$\begin{aligned} \epsilon Q J_+(z) &= -D_+^{[z]} \left(\frac{1}{z} \epsilon \lambda_3 + z \epsilon \lambda_1 \right) + \left(z - \frac{1}{z^3} \right) (D_{0+} \epsilon \lambda_1 - [N_+, \epsilon \lambda_1]) + \\ &\quad + \left(1 - \frac{1}{z^4} \right) [w_{1+}^*, \epsilon \lambda_3] \end{aligned} \quad (62)$$

$$\epsilon Q J_-(z) = -D_-^{[z]} \left(\frac{1}{z} \epsilon \lambda_3 + z \epsilon \lambda_1 \right) + \left(\frac{1}{z} - z^3 \right) (D_{0-} \epsilon \lambda_3 - [N_-, \epsilon \lambda_3]) + \quad (63)$$

$$+ (1 - z^4) [w_{3-}^*, \epsilon \lambda_1] \quad (64)$$

This means that:

$$\epsilon Q \hat{j} = d(g^{-1}(\epsilon \lambda_3 - \epsilon \lambda_1)g) \quad (65)$$

3 Vertex corresponding to β -deformation

3.1 The vertex and its descent

As discussed in Section 2.2, we introduce a set of formal anticommuting constants $\epsilon, \epsilon', \epsilon'', \dots$

Definition of the vertex It was proposed in [3] that the unintegrated vertex corresponding to the β -deformation is given by this expression:

$$V_{ab}^{\text{beta}}(\epsilon, \epsilon') = (g^{-1}\epsilon(\lambda_3 - \lambda_1)g)_a (g^{-1}\epsilon'(\lambda_3 - \lambda_1)g)_b \quad (66)$$

Here the indices a and b enumerate the adjoint representation of $psu(2, 2|4)$. Notice that (66) is antisymmetric under the exchange of a and b . Therefore this vertex is in the antisymmetric product of two adjoint representations of $psu(2, 2|4)$. We will parametrize the β -deformations by a constant antisymmetric tensor B^{ab} :

$$V[B](\epsilon, \epsilon') = B^{ab} (g^{-1}\epsilon(\lambda_3 - \lambda_1)g)_a (g^{-1}\epsilon'(\lambda_3 - \lambda_1)g)_b \quad (67)$$

Equivalence relation The antisymmetric product of two adjoint representations is not an irreducible representation. In particular, it has a subspace consisting of B^{ab} of the form: $B^{ab} = f_c^{ab} A^c$. It turns out that such B corresponds to BRST exact vertices:

$$\begin{aligned} f_c^{ab} V_{ab}^{\text{beta}} &= [(g^{-1}\epsilon(\lambda_3 - \lambda_1)g), (g^{-1}\epsilon'(\lambda_3 - \lambda_1)g)]_c = \\ &= \epsilon Q_{BRST}(g^{-1}\epsilon'(\lambda_3 + \lambda_1)g)_c \end{aligned} \quad (68)$$

Therefore the tensors B^{ab} and $B^{ab} + f_c^{ab} A^c$ give the same β -deformation:

$$B^{ab} \simeq B^{ab} + f_c^{ab} A^c \quad (69)$$

We will explain in Section 3.2 that the gauge transformation (69) should be accompanied by the change of variables (field redefinition). This is because the corresponding integrated vertex is only invariant on-shell.

Descent procedure and integrated vertex The deformation of the action corresponding to (66) follows from the standard descent procedure. Let us denote:

$$\Lambda_a(\epsilon) = (g^{-1}\epsilon(\lambda_3 - \lambda_1)g)_a \quad (70)$$

The operator $\Lambda_a(\epsilon)$ corresponds to the local conserved currents in the following sense:

$$d\Lambda_a(\epsilon) = \epsilon Q(j_a) \quad (71)$$

where $j_{a\pm}(\tau^+, \tau^-)$ is the density of the local conserved charge corresponding to the global symmetries. Therefore:

$$d(\Lambda_{[a}(\epsilon)\Lambda_{b]}(\epsilon')) = 2\epsilon Q j_{[a}\Lambda_{b]}(\epsilon') \quad (72)$$

and:

$$d(j_{[a}\Lambda_{b]}(\epsilon)) = -\frac{1}{2}\epsilon Q(j_{[a} \wedge j_{b]}) \quad (73)$$

We conclude that for any constant antisymmetric matrix B^{ab} we can infinitesimally deform the worldsheet action as follows:

$$S \rightarrow S + \frac{1}{2}B^{ab} \int j_{[a} \wedge j_{b]} \quad (74)$$

Summary of the descent procedure:

$$(d + Q) \left(V_1^{(0)}[B](\epsilon, \epsilon') + \epsilon V_1^{(1)}[B](\epsilon') + \epsilon \epsilon' V_1^{(2)}[B] \right) = 0 \quad (75)$$

where

$$\begin{aligned} V_1^{(0)}[B](\epsilon, \epsilon') &= \frac{1}{2}B^{ab} (g^{-1}\epsilon(\lambda_3 - \lambda_1)g)_a (g^{-1}\epsilon'(\lambda_3 - \lambda_1)g)_b \\ V_1^{(1)}[B](\epsilon') &= B^{ab} j_a (g^{-1}\epsilon'(\lambda_3 - \lambda_1)g)_b \\ V_1^{(2)}[B] &= \frac{1}{2}B^{ab} j_a \wedge j_b \end{aligned} \quad (76)$$

In Eq. (75) we assume that d commutes with ϵ .

“Bosonic” example Consider for example B^{ab} in the directions of S^5 . We get:

$$S \rightarrow S + B^{[kl][mn]} \left(\int X_{[k} dX_{l]} \wedge X_{[m} dX_{n]} + \dots \right) \quad (77)$$

where X_j describes the embedding of S^5 into \mathbf{R}^6 :

$$X_1^2 + X_2^2 + \dots + X_6^2 = 1 \quad (78)$$

and dots denote θ -dependent terms. These θ -dependent terms appear because j_a includes θ . The subspace $\mathbf{g} \subset \mathbf{g} \wedge \mathbf{g}$ corresponds to B of the following form:

$$B^{[kl][mn]} = \delta^{km} A^{ln} - \delta^{lm} A^{kn} + \delta^{ln} A^{km} - \delta^{kn} A^{lm} \quad (79)$$

where A^{mn} is antisymmetric matrix; then the corresponding deformation of the Lagrangian is a total derivative $d(A^{mn} X_m dX_n)$. The complementary space has real dimension 90, it corresponds to the representation $\mathbf{45_C}$ of $so(6)$.

3.2 What happens to the integrated vertex when B^{ab} is proportional to $f^{ab}_c A^c$?

In this case the integrated vertex becomes a total derivative Indeed, consider the descent procedure. When B^{ab} is proportional to the structure constant, this means that the vertex operator is of the form $[g^{-1}\epsilon\frac{d\lambda}{dl}g, g^{-1}\epsilon'\frac{d\lambda}{dl}g]$. Here we use:

$$l = \log z \quad (80)$$

— see Section 2.1.

We want to apply the descent procedure and obtain the corresponding integrated vertex operator. The first step is to take the derivative of our unintegrated vertex and see that it is BRST exact:

$$d \left[g^{-1}\epsilon\frac{d\lambda}{dl}g, g^{-1}\epsilon'\frac{d\lambda}{dl}g \right] = -2\epsilon Q \left[g^{-1}\frac{dJ}{dl}g, g^{-1}\epsilon'\frac{d\lambda}{dl}g \right] \quad (81)$$

But now a special thing happens; on the right hand side Q is taken of the expression which is d of something plus Q of something:

$$-2 \left[g^{-1}\frac{dJ}{dl}g, g^{-1}\epsilon'\frac{d\lambda}{dl}g \right] = \epsilon Q \left(g^{-1}\frac{d^2J}{dl^2}g \right) + d(g^{-1}\epsilon\lambda g)$$

This formula can be derived as follows:

$$\begin{aligned} \epsilon Q \left(g^{-1}\frac{d^2J}{dl^2}g \right) &= g^{-1} \left[\frac{d^2J}{dl^2}, \epsilon\lambda \right] g - g^{-1}\frac{d^2}{dl^2}(D\epsilon\lambda)g = \\ &= -2g^{-1} \left[\frac{dJ}{dl}, \frac{d\epsilon\lambda}{dl} \right] g - d(g^{-1}\epsilon\lambda g) \end{aligned} \quad (82)$$

The second (and the last) step of the descent procedure is to take the d of $-2 \left[g^{-1}\frac{dJ}{dl}g, g^{-1}\epsilon'\frac{d\lambda}{dl}g \right]$ and see that it is Q of some expression, which is then the corresponding integrated vertex operator. But we can see directly from (82) that in fact d of $-2 \left[g^{-1}\frac{dJ}{dl}g, g^{-1}\epsilon'\frac{d\lambda}{dl}g \right]$ is equal to Q of $d \left(g^{-1}\frac{d^2J}{dl^2}g \right)$. This means that, indeed, the corresponding integrated operator is a total derivative.

This can be easily seen explicitly. The integrated vertex operator is $[j\wedge, j] = -g^{-1} \left[\frac{dJ}{dl}\wedge, \frac{dJ}{dl} \right] g$. We observe:

$$\begin{aligned} d \left(g^{-1}\frac{d^2J}{dl^2}g \right) &= g^{-1}D\frac{d^2J}{dl^2}g = \\ &= -g^{-1} \left[\frac{dJ}{dl}\wedge, \frac{dJ}{dl} \right] g + \frac{1}{2}g^{-1} \left(\frac{d^2}{dl^2}DJ \right) g \end{aligned} \quad (83)$$

Notice that the second term on the right hand side $\frac{1}{2}g^{-1} \left(\frac{d^2}{dl^2}DJ \right) g$ is proportional to the equations of motion. Therefore, this term should be canceled by an infinitesimal field redefinition.

Field redefinition More precisely, $\text{Str} \left(t_a g^{-1} \left(\frac{d^2}{dt^2} D J \right) g \right)$ is the result of the variation of the action with respect to the infinitesimal left shift of g by $8(gt_a g^{-1})_1 - 8(gt_a g^{-1})_3$, plus some variation of λ and w . Let us denote this vector field \mathcal{X}_a :

$$\mathcal{X}_a S = \text{Str} \left(t_a g^{-1} \left(\frac{d^2}{dt^2} D J \right) g \right) \quad (84)$$

(where S is the action). We will not need the explicit form of \mathcal{X}_a in this paper. We conclude that:

- the infinitesimal gauge transformation $B^{ab} \rightarrow B^{ab} + f^{ab}{}_c A^c$ changes the vertex by a total derivative plus terms which can be absorbed into an infinitesimal field transformation corresponding to the vector field $\mathcal{X}_a A^a$

4 Constraint on the internal commutator

4.1 Additional constraint on B

The vertex (67) cannot as such be the right description of the β -deformation because it gives extra states which are not present in the supergravity description. For example, consider B of the form:

$$B^{ab} = \begin{cases} f_c^{ab} A^c & \text{if both } a \text{ and } b \text{ are even (bosonic) indices} \\ 0 & \text{otherwise} \end{cases} \quad (85)$$

where $A \in so(6) \subset \mathfrak{psu}(2, 2|4)$. The corresponding linearized excitation of $AdS_5 \times S^5$ is constant in the AdS directions, and transforms in the adjoint representation of $so(6)$ (rotations of S^5). But there is no such state in the supergravity spectrum [22].

Conjecture It is necessary for the consistency of the deformed worldsheet theory that the vertex $V^{(0)}$ is given by a primary operator. This condition was investigated in [23]; it was found that the double pole of the vertex operator with the energy-momentum tensor is proportional to the action of the Laplacian on $psu(2, 2|4)$. In our case, when $V^{(0)} = B^{ab} \Lambda_a \Lambda_b$, this is proportional to:

$$B^{ab} f_{ab}{}^c f_c{}^{de} \Lambda_d \Lambda_e = Q(B^{ab} f_{ab}{}^c \bar{\Lambda}_c) \quad (86)$$

Therefore if $B^{ab} f_{ab}{}^c \neq 0$ then the unintegrated vertex operator is not a conformal primary of the weight zero. Therefore we must impose this condition on B :

$$B^{ab} f_{ab}{}^c = 0 \quad (87)$$

The descent of the anomalous dimension The anomalous dimension of $V^{(0)}$ is Q_0 -exact:

$$\Delta V^{(0)} = Q_0 U^{(0)} \quad (88)$$

Let us act on this by d , and then use that $dV^{(0)} = Q_0 V^{(1)}$:

$$(\Delta - 1)dV^{(0)} = Q_0 dU^{(0)} \quad (89)$$

$$(\Delta - 1)Q_0 V^{(1)} = Q_0 dU^{(0)} \quad (90)$$

Then use that there is no Q_0 -cohomology in the conformal dimension 1 and ghost number 1. Therefore exists $U^{(1)}$ such that:

$$(\Delta - 1)V^{(1)} = dU^{(0)} + QU^{(1)} \quad (91)$$

$$(\Delta - 2)dV^{(1)} = QdU^{(1)} \quad (92)$$

$$(\Delta - 2)V^{(2)} = dU^{(1)} \quad (93)$$

Therefore:

$$\left[\begin{array}{c} \text{anomalous dimension} \\ \text{of the unintegrated vertex} \\ \text{is BRST exact} \end{array} \right] \Rightarrow \left[\begin{array}{c} \text{anomalous dimension} \\ \text{of the integrated vertex} \\ \text{is a total derivative} \end{array} \right] \quad (94)$$

In our case Eq. (91) is $\log \epsilon$ times Eq. (82); *i.e.* $U^{(1)}$ is proportional to $g^{-1} \frac{d^2 J}{dI^2} g$.

Renormalization of the integrated vertex We can demonstrate that the integrated vertex has a nonzero anomalous dimension in the case when $B^{ab} f_{ab}^c \neq 0$. Let us pick a point in $AdS_5 \times S^5$ and consider the near flat space expansion around this fixed point as in [20]. This means that we write $g = e^{\vartheta/R} e^{x/R}$ and expand around the selected point $x = \vartheta = 0$. Suppose that the only nonzero components of B are in $\mathfrak{g}_0 \wedge \mathfrak{g}_0$ where \mathfrak{g}_0 corresponds to the rotations around the selected point. In other words B is $B^{[\mu\nu][\rho\sigma]}$. Then the integrated vertex $V^{(2)}$ contains the terms:

$$B^{[\mu\nu][\rho\sigma]} x_\mu dx_\nu \wedge x_\rho dx_\sigma \quad (95)$$

The log divergence comes from the contraction of x_μ and x_ρ :

$$\log \epsilon g_{\mu\rho} B^{[\mu\nu][\rho\sigma]} dx_\nu \wedge dx_\sigma \quad (96)$$

This is indeed proportional to $B^{ab} f_{ab}^c$.

4.2 Example of an unintegrated vertex violating the constraint

Pick a constant $A \in \mathfrak{g}_2$ and consider the following vertex operators:

$$V_A(\epsilon, \epsilon') = \text{Str} \left(A \left[(g^{-1}(\epsilon\lambda_3 - \epsilon\lambda_1)g)_2, (g^{-1}(\epsilon'\lambda_3 - \epsilon'\lambda_1)g)_0 \right] \right) \quad (97)$$

This operator corresponds to the following B^{ab} :

$$B^{ab} = \begin{cases} f^{ab}{}_c A^c & \text{for } a \text{ in } \mathfrak{g}_2 \text{ and } b \text{ in } \mathfrak{g}_0 \\ & \text{or } a \text{ in } \mathfrak{g}_2 \text{ and } b \text{ in } \mathfrak{g}_0 \\ 0 & \text{otherwise} \end{cases} \quad (98)$$

The internal commutator is:

$$B^{ab} f_{ab}{}^c = C_{so(6)} A^c \neq 0 \quad (99)$$

where $C_{so(6)}$ is the adjoint Casimir of $so(6)$. Therefore the internal commutator constraint is not satisfied for this vertex. There is no such state in Type IIB SUGRA on $AdS_5 \times S^5$.

4.3 What happens in the flat space limit

In order to better understand this vertex we will consider its flat space limit. We will use the flat space expansion similar to the one used in [20]. We will write:

$$g = e^{x_2 + \theta_3 + \theta_1} \quad (100)$$

and consider x and θ small. To reproduce the flat space BRST operator we consider the “flat space scaling”:

$$x \simeq R^{-2}, \quad \theta \simeq R^{-1}, \quad \lambda \simeq R^{-1} \quad (101)$$

R is the radius of the AdS space entering the action as in (44).

Notice that there are two differences with [20]; [20] used a different gauge $g = e^\theta e^x$; also that paper used the “uniform” scaling $x \simeq \theta \simeq \lambda \simeq R^{-1}$ which is different from the “flat space” scaling (101) which we use here. In the flat space limit the “flat space scaling” gives the BRST operator $\lambda^\alpha \left(\frac{\partial}{\partial \theta^\alpha} + \Gamma_{\alpha\beta}^m \theta^\beta \frac{\partial}{\partial x^m} \right)$, which is the correct BRST operator in flat space. While the “uniform” scaling $x \simeq \theta \simeq \lambda \simeq R^{-1}$ used in [20] gives $\lambda^\alpha \frac{\partial}{\partial \theta^\alpha}$.

With these notations the vertex operator becomes a function of x, θ, λ . The BRST operator in terms of x, θ, λ is calculated in Appendix B. The expansion of the vertex (97) starts with the following terms:

$$V_A(\epsilon, \epsilon') = \text{Str} \left(A \left([[\theta_3, \epsilon \lambda_3], [\theta_1, \epsilon' \lambda_3]] + [[\theta_1, \epsilon \lambda_1], [\theta_3, \epsilon' \lambda_1]] - \right. \right. \\ \left. \left. - [[\theta_3, \epsilon \lambda_1], [\theta_3, \epsilon' \lambda_3]] - [[\theta_1, \epsilon \lambda_1], [\theta_1, \epsilon' \lambda_3]] + \dots \right) \right) \quad (102)$$

where \dots stands for the terms of higher order in $1/R$ expansion.

We used a Mathematica program to recast $V_A(\epsilon, \epsilon')$ in various BRST-equivalent forms. It turns out that $V_A(\epsilon, \epsilon')$ is BRST equivalent to the following expression:

$$\text{Str} \left(A \left(-\frac{8}{9} [[\theta_3, [\theta_3, \epsilon \lambda_3]], [\theta_3, [\theta_3, \epsilon' \lambda_3]]] - \right. \right. \\ \left. \left. - \frac{8}{9} [[\theta_1, [\theta_1, \epsilon \lambda_1]], [\theta_1, [\theta_1, \epsilon' \lambda_1]]] + \dots \right) \right) \quad (103)$$

On the other hand, $V_A(\epsilon, \epsilon')$ is also equivalent to this:

$$\begin{aligned} \text{Str} \Big(& A \Big(4 [\ x \ , \ [[\theta_1, \epsilon \lambda_1] \ , \ [\theta_3, \epsilon' \lambda_3] \]] - \\ & - [[\theta_1, \epsilon \lambda_1] \ , \ [\theta_3, [\theta_3, [\theta_3, \epsilon' \lambda_3]]] \] - \\ & - [[\theta_3, \epsilon \lambda_3] \ , \ [\theta_1, [\theta_1, [\theta_1, \epsilon' \lambda_1]]] \] + \dots \Big) \Big) \end{aligned} \quad (104)$$

In both (103) and (104) ... stands for terms of the order R^{-8} and higher in $1/R$ expansion.

We will call (103) “the (2,0)+(0,2)-gauge” and (104) “the (1,1)-gauge”.

Flat space notations Eqs. (103) and (104) are written in terms of the algebraic structures of $psu(2, 2|4)$, the commutator and the supertrace. It is possible to rewrite them using the gamma-matrices.

The (2,0)+(0,2) gauge expression (103) reads:

$$\begin{aligned} & -\frac{8}{9}(\theta_3 \Gamma_{klm} \theta_3)(\theta_3 \Gamma^k \epsilon \lambda_3) \overline{A}^l (\theta_3 \Gamma^m \epsilon' \lambda_3) - \\ & -\frac{8}{9}(\theta_1 \Gamma_{klm} \theta_1)(\theta_1 \Gamma^k \epsilon \lambda_1) \overline{A}^l (\theta_1 \Gamma^m \epsilon' \lambda_1) + \dots \end{aligned} \quad (105)$$

where we denoted:

$$\overline{A}^l = \begin{cases} A^l & \text{if } l \in \{0, \dots, 4\} \\ -A^l & \text{if } l \in \{5, \dots, 9\} \end{cases} \quad (106)$$

On the other hand, the (1,1) expression (104) reads:

$$\begin{aligned} & 2 (\overline{A}_{[m} x_{n]} + A_{[m} \overline{x}_{n]}) (\theta_1 \Gamma^m \epsilon \lambda_1)(\theta_3 \Gamma^n \epsilon' \lambda_3) - \\ & - A_n (\theta_1 \Gamma_m \epsilon \lambda_1)(\theta_3 \overline{\Gamma}^{[m} \Gamma^{n]} \Gamma^l \theta_3)(\theta_3 \Gamma_l \epsilon' \lambda_3) - \\ & - A_n (\theta_3 \Gamma_m \epsilon \lambda_3)(\theta_1 \overline{\Gamma}^{[m} \Gamma^{n]} \Gamma^l \theta_1)(\theta_1 \Gamma_l \epsilon' \lambda_1) \end{aligned} \quad (107)$$

We leave the target-space interpretation of these states (even in flat space) as an open question.

5 Symmetries of the vertex

In this section we will collect the necessary fact from the representation theory and discuss the symmetries of our vertex.

5.1 Representation theory

We will denote:

$$\mathbf{g} = psu(2, 2|4) \quad (108)$$

$$\widehat{\mathbf{g}} = su(2, 2|4) \quad (109)$$

$$\widehat{\mathbf{g}}' = u(2, 2|4) \quad (110)$$

5.1.1 Matrix notations

Elements of $\widehat{\mathbf{g}}'$ are block matrices:

$$u_b^a = \begin{pmatrix} u_j^i & u_\alpha^i \\ u_i^\alpha & u_\beta^\alpha \end{pmatrix} \quad (111)$$

satisfying some hermiticity property. The precise form of the hermiticity property will not be important for us. The indices i, j, k, \dots correspond to the fundamental representation of the $su(4)$ and the indices $\alpha, \beta, \gamma, \dots$ to the fundamental of the $su(2, 2)$. The letters a, b, c, \dots stand for either i, j, k, \dots or $\alpha, \beta, \gamma, \dots$.

5.1.2 Exterior product of two adjoint representations

Let us consider the exterior product of two adjoint representations of $\widehat{\mathbf{g}}'$:

$$\widehat{\mathbf{g}}' \wedge \widehat{\mathbf{g}}' \quad (112)$$

In matrix notations, this is the space of matrices b_{bd}^{ac} satisfying the antisymmetry property:

$$b_{bd}^{ac} = (-)^{(\bar{a}+\bar{b})(\bar{c}+\bar{d})+1} b_{db}^{ca} \quad (113)$$

where \bar{I} is 0 if I is the index of $su(2, 2)$ and 1 if I is the index of $su(4)$.

The difference between \mathbf{g} and $\widehat{\mathbf{g}}'$ is in the central charge c and the differentiation s . The central charge is the unit 8×8 matrix, and the differentiation is $\text{diag}(1, 1, 1, 1, -1, -1, -1, -1)$.

Let \mathbf{R}_s denote the 1-dimensional linear space spanned by the differentiation and \mathbf{R}_c the 1-dimensional linear space spanned by the central charge. Let us consider $\widehat{\mathbf{g}}' \wedge \widehat{\mathbf{g}}'$ as a representation of \mathbf{g} . We observe that $\widehat{\mathbf{g}}' = \mathbf{g} + \mathbf{R}_s + \mathbf{R}_c$. Therefore:

$$\widehat{\mathbf{g}}' \wedge \widehat{\mathbf{g}}' = \mathbf{g} \wedge \mathbf{g} + \mathbf{R}_s \otimes \mathbf{g} + \mathbf{R}_c \otimes \mathbf{g} + \mathbf{R}_s \otimes \mathbf{R}_c \quad (114)$$

We observe the following facts about $\mathbf{g} \wedge \mathbf{g}$:

1. The representation $\mathbf{g} \wedge \mathbf{g}$ is not irreducible, because it contains two invariant subspaces:

$$(\mathbf{g} \wedge \mathbf{g})_0 \text{ consisting of } \sum_I x_I \wedge y_I \text{ such that } \sum_I [x_I, y_I] = 0 \quad (115)$$

$$\mathbf{g} \subset (\mathbf{g} \wedge \mathbf{g})_0 \text{ spanned by } f_a^{bc} t_b \wedge t_c \quad (116)$$

2. We therefore have two exact sequences:

$$0 \rightarrow \mathfrak{g} \rightarrow (\mathfrak{g} \wedge \mathfrak{g})_0 \rightarrow (\mathfrak{g} \wedge \mathfrak{g})_0/\mathfrak{g} \rightarrow 0 \quad (117)$$

$$0 \rightarrow (\mathfrak{g} \wedge \mathfrak{g})_0 \rightarrow (\mathfrak{g} \wedge \mathfrak{g}) \rightarrow \mathfrak{g} \rightarrow 0 \quad (118)$$

Both of them *do not split*. This means that there is no complementary subspace to (116) in $(\mathfrak{g} \wedge \mathfrak{g})_0$ and no complementary subspace to (115) in $\mathfrak{g} \wedge \mathfrak{g}$.

We introduce an “inner commutator” map F :

$$F : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}, \quad F\left(\sum_I x_I \wedge y_I\right) = \sum_I [x_I, y_I] \quad (119)$$

With this notation $(\mathfrak{g} \wedge \mathfrak{g})_0 = \text{Ker } F$. Notice that $(x, y) = \text{Str}(xy)$ is a nondegenerate symmetric scalar product, but this scalar product is not positive definite. Let F^* denote the conjugate to F with respect to this scalar product. Because the Casimir operators vanish in the adjoint representation we have:

$$FF^* = 0 \quad (120)$$

On the other hand,

$$F^*F : \mathfrak{g} \wedge \mathfrak{g} \rightarrow (\mathfrak{g} \wedge \mathfrak{g})_0 \quad (121)$$

is non-zero. In fact F^*F can be identified with the action of the quadratic Casimir of $\mathfrak{psu}(2, 2|4)$ on $\mathfrak{g} \wedge \mathfrak{g}$:

$$\Delta_2 = C^{ab}t_a t_b = F^*F \quad (122)$$

Note that Δ_2 on $\mathfrak{g} \wedge \mathfrak{g}$ is nilpotent: $(\Delta_2)^2 = 0$. We conclude that:

- The space of linearized β -deformations $(\mathfrak{g} \wedge \mathfrak{g})_0/\mathfrak{g}$ can be identified with $\frac{\text{Ker } \Delta_2}{\text{Im } \Delta_2}$.

5.1.3 Complex structure

Notice that $\widehat{\mathfrak{g}}' \wedge \widehat{\mathfrak{g}}'$ has a complex structure, which acts as a multiplication by i and the exchange of the upper indices:

$$\mathcal{I} b_{JL}^{IK} = i b_{JL}^{KI} \quad (123)$$

Notice that $\mathcal{I}^2 = -1$. Let us discuss the action of \mathcal{I} on the decomposition (114). We get:

$$\mathcal{I}(\mathbf{R}_c \otimes \mathfrak{g}) = (\mathfrak{g} \subset \mathfrak{g} \wedge \mathfrak{g}) \quad (124)$$

$$\mathcal{I}(\mathfrak{g} \subset \mathfrak{g} \wedge \mathfrak{g}) = (\mathbf{R}_c \otimes \mathfrak{g}) \quad (125)$$

Generally speaking $\mathcal{I}(x \wedge y)$ has a component in $\mathbf{R}_s \otimes \mathfrak{g}$, but when restricted on $(\mathfrak{g} \wedge \mathfrak{g})_0$ it lands into $(\mathfrak{g} \wedge \mathfrak{g})_0 + \mathbf{R}_c \otimes \mathfrak{g}$. We conclude that:

- the operation \mathcal{I} induces a complex structure on the space of linearized β -deformations $(\mathfrak{g} \wedge \mathfrak{g})_0/\mathfrak{g}$

5.2 Our vertex is not covariant

The linearized β -deformations transform in the following representation of $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$:

$$(\mathfrak{g} \wedge \mathfrak{g})_0 / \mathfrak{g} \quad (126)$$

But the B tensor satisfying $B^{ab} f_{ab}{}^c = 0$ is in $(\mathfrak{g} \wedge \mathfrak{g})_0$. The object transforming in $(\mathfrak{g} \wedge \mathfrak{g})_0 / \mathfrak{g}$ is the equivalence class of $B^{ab} \simeq B^{ab} + f^{ab}{}_c A^c$. Let us denote this equivalence class $[B]$. The short exact sequence (117) does not split. Therefore it is not possible to pick a representative for B in the equivalence class $[B]$ in a way consistent with the supersymmetry.

It was argued in [3] that there is always a way to choose the vertex covariantly, *i.e.* in a way consistent with the supersymmetry. However the proof used the assumption that the representation of \mathfrak{g} in which the state transforms has a sufficiently large spin when restricted to $so(6) \subset \mathfrak{g}$. The β -deformation is a low-spin case, so there is an obstacle to choosing the vertex in a covariant way.

6 General deformation theory

6.1 Some general notations

We will consider the deformation of the Lagrangian of the following form:

$$\mathcal{L}_{deformed}^{(2)} = \mathcal{L}^{(2)} + \varepsilon V_1^{(2)} + \varepsilon^2 V_2^{(2)} + \dots \quad (127)$$

Here the upper index $^{(2)}$ indicates that the object is a 2-form (*e.g.* the action density $\mathcal{L}^{(2)}$). When we say that some equation is valid “on-shell” we will generally speaking mean on-shell with respect to the undeformed Lagrangian $\mathcal{L}^{(2)}$. We will write:

$$F \simeq 0 \quad (128)$$

when F is zero up to the equations of motion of $\mathcal{L}^{(2)}$. Given some infinitesimal field transformation ξ we will define $I_\xi^{(1)}$ by the following formula:

$$\xi \cdot \mathcal{L}^{(2)} \simeq dI_\xi^{(1)} \quad (129)$$

This equation holds on-shell, but we will consider it in situations where it actually defines $I_\xi^{(1)}$ also off-shell. The only ambiguity would be to add to $I_\xi^{(1)}$ some local conserved current, but for those ξ which we need there will be no local conserved currents with appropriate symmetries. We conclude that:

- for every vector field ξ there is a 1-form $I_\xi^{(1)}$ defined by (129); it is defined up to d of something

6.2 Deforming with integrated vertex operator

Let us return to the descent procedure discussed in Section 3.1 Eq. (75). The general relation is:

$$(d + Q) (V^{(0)} + V^{(1)} + V^{(2)}) = 0 \quad (130)$$

Given the vertex operator V , we can perturb the action by adding to it $\int V^{(2)}$:

$$\int [dg \, d\lambda \, dw] e^{\int \mathcal{L}^{(2)}} \longrightarrow \int [dg \, d\lambda \, dw] e^{\int (\mathcal{L}^{(2)} + \varepsilon V_1^{(2)} + \dots)} \quad (131)$$

where ε is an infinitesimally small parameter. (The lower index 1 in $V_1^{(2)}$ is to indicate that $V_1^{(2)}$ is the coefficient of the first power of ε .)

6.2.1 BRST invariance at the first order in ε

The vertex operator $V_1^{(2)}$ in (131) should be such that *on-shell* $Q_0 V_1^{(2)}$ is a total derivative. Generally speaking this means that there exists an odd vector field, which we will call Q_1 , and a 1-form $X_1^{(1)}$ such that:

$$Q_0 V_1^{(2)} + Q_1 \mathcal{L}^{(2)} = dX_1^{(1)} \quad (132)$$

Comment 1: Notice that Eq. (132) determines Q_1 up to an infinitesimal transformation of ghost number one which leaves the action invariant. We assume that there are no such infinitesimal transformations except for Q_0 (in other words the BRST symmetry is the only symmetry with the ghost number one). Under this assumption the infinitesimal transformation Q_1 is defined by (132) unambiguously.

Comment 2: The combined $Q_0 + \varepsilon Q_1$ is a symmetry of the action off-shell. But is this a nilpotent symmetry? In other words, is it true that $\{Q_0, Q_1\} = 0$? In fact this is true, for the following reason. Observe that if $\{Q_0, Q_1\}$ is not zero, then it would be a symmetry of the unperturbed action:

$$\{Q_0, Q_1\} S_0 = -Q_0^2 V^{(2)} = 0 \quad (133)$$

Under the assumption that there are no symmetries of the ghost number 2, we conclude that $\{Q_0, Q_1\}$ should be zero⁷. Therefore $Q_0 + \varepsilon Q_1$ is automatically nilpotent.

6.2.2 BRST invariance at the second order in ε

Eq. (132) guarantees that the deformation (131) exists at the first order in ε . Similarly, the consistency condition at the second order in ε is:

$$Q_1 V_1^{(2)} + Q_0 V_2^{(2)} + Q_2 \mathcal{L}^{(2)} = dX_2^{(1)} \quad (134)$$

⁷See footnote on p. 7 of [19]. A.M. would like to thank V. Puletti for a discussion about this.

This equation is a definition of Q_2 and $V_2^{(2)}$; the existence of Q_2 and $V_2^{(2)}$ satisfying (134) is the consistency condition. But it is more convenient to describe the consistency condition in terms of the dimension zero operators (unintegrated vertices). We will now translate the consistency condition (134) from the dimension two language to the dimension zero language.

6.2.3 The descent of $Q_1 V_1^{(2)}$

Let us act on (132) with Q_1 . We get⁸:

$$-Q_0 Q_1 V_1^{(2)} \simeq d(Q_1 X_1^{(1)} - I_{Q_1^2}^{(1)}) \quad (135)$$

Therefore $d(Q_0 Q_1 X_1^{(1)} - Q_0 I_{Q_1^2}^{(1)}) \simeq 0$. In fact this is also true off-shell because there are no local conserved charges of the ghost number three:

$$Q_0(Q_1 X_1^{(1)} - I_{Q_1^2}^{(1)}) = dW_2^{(0)} \quad (136)$$

This equation is the definition of $W_2^{(0)}$. An alternative notation for $W_2^{(0)}$ could be $-(Q_1 V_1^{(2)})^{(0)}$.

Condition on $W_2^{(0)}$ Now we want to derive a constraint on $W_2^{(0)}$ following from (134). On the left hand side of (135), let us replace:

$$Q_1 V_1^{(2)} \rightarrow dX_2^{(1)} - Q_0 V_2^{(2)} - Q_2 \mathcal{L}$$

as follows from (134), and use $Q_2 \mathcal{L} \simeq dI_{Q_2}^{(1)}$. We get:

$$d(-Q_0 X_2^{(1)} + Q_0 I_{Q_2}^{(1)}) \simeq d(Q_1 X_1^{(1)} - I_{Q_1^2}^{(1)}) \quad (137)$$

This implies:

$$-Q_0 X_2^{(1)} + Q_0 I_{Q_2}^{(1)} = Q_1 X_1^{(1)} - I_{Q_1^2}^{(1)} + d(\text{smth}) \quad (138)$$

Therefore in this case $W_2^{(0)}$ defined by (136) is Q_0 -exact:

$$W_2^{(0)} = Q_0(\text{smth}) \quad (139)$$

6.2.4 Going back

Now suppose that (139) is satisfied:

$$W_2^{(0)} = Q_0 T_2^{(0)}$$

Then we get:

$$Q_0(Q_1 X_1^{(1)} - I_{Q_1^2}^{(1)} - dT_2^{(0)}) = 0 \quad (140)$$

Let us assume that the following is true:

⁸The only reason why this equation would not hold off-shell is the use of $Q_1^2 \mathcal{L} = dI_{Q_1^2}^{(1)}$.

- the cohomology of Q_0 on 1-forms of the ghost number 2 is trivial

Then (140) implies the existence of $X_2^{(1)}$ such that:

$$Q_1 X_1^{(1)} - I_{Q_1^2}^{(1)} = dT_2^{(0)} - Q_0 X_2^{(1)} \quad (141)$$

Let us compare this to (135): $-Q_0 Q_1 V_1^{(2)} \simeq d(Q_1 X_1^{(1)} - I_{Q_1^2}^{(1)})$. We get:

$$Q_0(Q_1 V_1^{(2)} - dX_2^{(1)}) \simeq 0 \quad (142)$$

Let us assume that:

- the covariant cohomology of Q_0 on 2-forms of the ghost number 1 is trivial

Then (142) implies the existence of $V_2^{(2)}$ and Q_2 such that (134). This means that (139) is the necessary and sufficient condition for the deformation to exist at the second order in ε .

6.3 Comment about Q_2

We will see in the next section that in our particular case (the beta-deformation) $W_2^{(0)} = 0$ implies $Q_1^2 = 0$. This implies that $Q_2 = 0$ (because Eq. (142) holds true off-shell; see the footnote before Eq. (135)).

6.4 Higher orders of perturbation theory

6.4.1 Going forward (necessary condition)

Suppose that we have identified $V_p^{(2)}$ and Q_p up to the order n , *i.e.* for $p = 1, 2, \dots, n$, so that:

$$\begin{aligned} Q_0 V_p^{(2)} + Q_1 V_{p-1}^{(2)} + \dots + Q_{p-1} V_1^{(2)} + Q_p \mathcal{L}^{(2)} &= dX_p^{(1)} \\ Q_0 Q_p + Q_1 Q_{p-1} + \dots + Q_{p-1} Q_1 + Q_p Q_0 &= 0 \end{aligned} \quad (143)$$

Then:

$$\begin{aligned} &Q_1 Q_0 V_n^{(2)} + Q_1 Q_1 V_{n-1}^{(2)} + Q_1 Q_2 V_{n-2}^{(2)} + \dots + Q_1 Q_{n-1} V_1^{(2)} + Q_1 Q_n \mathcal{L}^{(2)} + \\ &+ Q_2 Q_0 V_{n-1}^{(2)} + Q_2 Q_1 V_{n-2}^{(2)} + \dots + Q_2 Q_{n-2} V_1^{(2)} + Q_2 Q_{n-1} \mathcal{L}^{(2)} + \\ &+ Q_3 Q_0 V_{n-2}^{(2)} + \dots + Q_3 Q_{n-3} V_1^{(2)} + Q_3 Q_{n-2} \mathcal{L}^{(2)} + \\ &\quad + \dots + \\ &\quad + Q_n Q_0 V_1^{(2)} + Q_n Q_1 \mathcal{L}^{(2)} \simeq \\ &\simeq -Q_0(Q_1 V_n^{(2)} + Q_2 V_{n-1}^{(2)} + \dots + Q_n V_1^{(2)}) + dI_{Q_1 Q_n + Q_2 Q_{n-1} + \dots + Q_n Q_1}^{(1)} \simeq \\ &\simeq d(Q_1 X_n^{(1)} + Q_2 X_{n-1}^{(1)} + \dots + Q_n X_1^{(1)}) \end{aligned} \quad (144)$$

This implies the existence of $W_{n+1}^{(0)}$ such that:

$$Q_0(Q_1X_n^{(1)} + \dots + Q_nX_1^{(1)} - I_{Q_1Q_n+\dots+Q_nQ_1}^{(1)}) = dW_{n+1}^{(0)} \quad (145)$$

In other words, the validity of (143) for $p \in \{1, \dots, n\}$ allows us to define $W_{n+1}^{(0)}$ by Eq. (145). Now, suppose that we can construct $V_{n+1}^{(2)}$ and Q_{n+1} , so that

$$Q_0V_{n+1}^{(2)} + Q_1V_n^{(2)} + \dots + Q_nV_1^{(2)} + Q_{n+1}\mathcal{L}^{(2)} = dX_{n+1}^{(1)} \quad (146)$$

Then this implies that $W_{n+1}^{(0)}$ satisfies some conditions. Indeed, applying Q_0 to (146) we get:

$$Q_0(Q_1V_n^{(2)} + \dots + Q_nV_1^{(2)}) \simeq dQ_0(X_{n+1}^{(1)} - I_{Q_{n+1}}^{(1)}) \quad (147)$$

Now, returning to (144) we derive:

$$\begin{aligned} d(Q_1X_n^{(1)} + Q_2X_{n-1}^{(1)} + \dots + Q_nX_1^{(1)} - I_{Q_1Q_n+Q_2Q_{n-1}+\dots+Q_nQ_1}^{(1)}) &= \\ &= -dQ_0(X_{n+1}^{(1)} - I_{Q_{n+1}}^{(1)}) \end{aligned} \quad (148)$$

and therefore there exists such a $T_{n+1}^{(0)}$ that:

$$\begin{aligned} Q_1X_n^{(1)} + Q_2X_{n-1}^{(1)} + \dots + Q_nX_1^{(1)} - I_{Q_1Q_n+Q_2Q_{n-1}+\dots+Q_nQ_1}^{(1)} &= \\ &= -Q_0(X_{n+1}^{(1)} - I_{Q_{n+1}}^{(1)}) + dT_{n+1}^{(0)} \end{aligned} \quad (149)$$

This implies:

$$W_{n+1}^{(0)} = Q_0T_{n+1}^{(0)} \quad (150)$$

6.4.2 Going back (sufficient condition)

Now suppose that (150) holds. Then

$$Q_0(Q_1X_n^{(1)} + \dots + Q_nX_1^{(1)} - I_{Q_1Q_n+\dots+Q_nQ_1}^{(1)} - dT_{n+1}^{(0)}) = 0 \quad (151)$$

Assuming the triviality of the cohomology of Q_0 on the 1-forms of the ghost number 2, we conclude the existence of $X_{n+1}^{(1)}$ such that:

$$Q_1X_n^{(1)} + \dots + Q_nX_1^{(1)} - I_{Q_1Q_n+\dots+Q_nQ_1}^{(1)} = dT_{n+1}^{(0)} - Q_0X_{n+1}^{(1)} \quad (152)$$

This and (144) implies, under the assumption that the cohomology of Q_0 on the 2-forms of the ghost number 1 is zero, the existence of $V_{n+1}^{(2)}$ such that:

$$Q_1V_n^{(2)} + Q_2V_{n-1}^{(2)} + \dots + Q_nV_1^{(2)} \simeq dX_{n+1}^{(1)} - Q_0V_{n+1}^{(2)} \quad (153)$$

This means that $W_{n+1}^{(0)}$ being Q_0 -exact is not only a necessary, but also a sufficient condition to be able to extend the deformation to the order $n + 1$. The off-shell version of Eq. (153):

$$Q_0 V_{n+1}^{(2)} + Q_1 V_n^{(2)} + Q_2 V_{n-1}^{(2)} + \dots + Q_n V_1^{(2)} + Q_{n+1} \mathcal{L} = dX_{n+1}^{(1)} \quad (154)$$

This is the definition of Q_{n+1} . Notice that so defined Q_{n+1} satisfies:

$$(Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots + \varepsilon^{n+1} Q_{n+1})^2 = O(\varepsilon^{n+2}) \quad (155)$$

The proof goes by induction. We start with $Q_0^2 = 0$. The induction hypothesis is $(Q_0 + \varepsilon Q_1 + \dots + \varepsilon^n Q_n)^2 = O(\varepsilon^{n+1})$, and this guarantees that $(Q_0 + \varepsilon Q_1 + \dots + \varepsilon^{n+1} Q_{n+1})^2$ is also at least as small as $O(\varepsilon^{n+1})$:

$$(Q_0 + \varepsilon Q_1 + \dots + \varepsilon^{n+1} Q_{n+1})^2 = O(\varepsilon^{n+1}) \quad (156)$$

By construction:

$$\int (Q_0 + \varepsilon Q_1 + \dots + \varepsilon^n Q_n + \varepsilon^{n+1} Q_{n+1})(\mathcal{L}^{(2)} + \varepsilon V_1^{(2)} + \dots + \varepsilon^{n+1} V_{n+1}^{(2)}) = O(\varepsilon^{n+2}) \quad (157)$$

Therefore:

$$\int (Q_0 + \varepsilon Q_1 + \dots + \varepsilon^n Q_n + \varepsilon^{n+1} Q_{n+1})^2 (\mathcal{L}^{(2)} + \varepsilon V_1^{(2)} + \dots + \varepsilon^{n+1} V_{n+1}^{(2)}) = O(\varepsilon^{n+2}) \quad (158)$$

This and (156) imply:

$$\int (Q_0 + \varepsilon Q_1 + \dots + \varepsilon^n Q_n + \varepsilon^{n+1} Q_{n+1})^2 \mathcal{L}^{(2)} = O(\varepsilon^{n+2}) \quad (159)$$

Let P_{n+1} denotes the coefficient of ε^{n+1} :

$$(Q_0 + \varepsilon Q_1 + \dots + \varepsilon^n Q_n + \varepsilon^{n+1} Q_{n+1})^2 = \varepsilon^{n+1} P_{n+1} + \dots \quad (160)$$

Then (159) implies that P_{n+1} is a symmetry of the undeformed theory: $\int P_{n+1} \mathcal{L} = 0$. Under the assumption that the pure spinor superstring in $AdS_5 \times S^5$ does not have conservation laws of the ghost number two, it follows that $P_{n+1} = 0$. This completes the step of the induction.

Conclusion Provided that the deformation of the action is defined up to the order n in ε , the obstacle to defining the deformation to the order $n + 1$ is the Q_0 cohomology class of $W_{n+1}^{(0)}$.

7 Applying the general theory to beta-deformation

7.1 Calculation of Q_1

The off-shell version of Eq. (73) for $QV_1^{(2)}$ is:

$$\epsilon Q \left(\frac{1}{2} B^{ab} j_a \wedge j_b \right) = B^{ab} d\Lambda_a(\epsilon) \wedge j_b \quad (161)$$

For the deformed action to be BRST invariant off-shell, we need to modify the BRST transformation:

$$Q = Q_0 + Q_1 \quad (162)$$

where Q_0 is the original (pure $AdS_5 \times S^5$) BRST transformation, and Q_1 is the modification.

We will now argue that Q_1 is in fact a $psu(2, 2|4)$ transformation with the *space-time-dependent parameter*. First of all, notice that under the global rotations:

$$g \rightarrow gg_0, \quad g_0 = \text{const} \quad (163)$$

the Lagrangian is invariant. But what will happen if we allow g_0 to depend on τ^\pm ? Consider an infinitesimal transformation:

$$\delta g = g\alpha \quad \text{where } \alpha = \alpha(\tau^+, \tau^-) \quad (164)$$

Then:

$$\delta J = -g d\alpha g^{-1} \quad (165)$$

and the variation of the Lagrangian is:

$$\begin{aligned} \delta \mathcal{L} &= \frac{1}{4} \text{Str} \left(g^{-1} \frac{dJ}{dl} g \wedge d\alpha \right) = \\ &= \frac{1}{4} \text{Str} \left[(\hat{j}_+ + 4g^{-1}w_{1+}^*g) \partial_- \alpha - (\hat{j}_- - 4g^{-1}w_{3-}^*g) \partial_+ \alpha \right] d\tau^+ \wedge d\tau^- \end{aligned} \quad (166)$$

In order to derive this formula, it is useful to rewrite the Lagrangian (44) in the following interesting form:

$$\begin{aligned} \mathcal{L} &= \text{Str} \left(\frac{1}{4} J_+ \frac{dJ_-}{dl} \Big|_{l=0} + w_{1+} \partial_- \lambda_3 + w_{3-} \partial_+ \lambda_1 - N_{0+} N_{0-} - w_{1+}^* w_{3-}^* \right) = \\ &= \text{Str} \left(-\frac{1}{4} \frac{dJ_+}{dl} \Big|_{l=0} J_- + w_{1+} \partial_- \lambda_3 + w_{3-} \partial_+ \lambda_1 - N_{0+} N_{0-} - w_{1+}^* w_{3-}^* \right) \end{aligned} \quad (167)$$

and then use (165).

We are now ready to calculate Q_1 . To start, let us consider the following infinitesimal transformation Ξ_α :

$$\Xi_\alpha : \quad \Xi_\alpha g = g\alpha, \quad \Xi_\alpha w_{3-}^* = \mathcal{P}_{31}(g \partial_- \alpha g^{-1})_3, \quad \Xi_\alpha w_{1+}^* = \mathcal{P}_{13}(g \partial_+ \alpha g^{-1})_1 \quad (168)$$

This transformation combines the “localized” rotation (164) with the shift of the antifields w^* . Here the projectors \mathcal{P}_{13} and \mathcal{P}_{31} are defined by the formulas:

$$\begin{aligned}\mathcal{P}_{13}A_1 &= A_1 + [\lambda_3, \text{smth}_2] \\ [\lambda_1, \mathcal{P}_{13}A_1] &= 0 \\ \mathcal{P}_{31}A_3 &= A_3 + [\lambda_1, \text{smth}_2] \\ [\lambda_3, \mathcal{P}_{31}A_3] &= 0\end{aligned}\tag{169}$$

The purpose of these projectors in (178) is to enforce the constraints (57).

On the definition of \mathcal{P} . In this paragraph we will prove the existence of the projector \mathcal{P}_{13} satisfying these properties. We will start with the following lemma:

Lemma 7.1.1: If $\{[S_2, \lambda_3], \lambda_1\} = 0$ then $[S_2, \lambda_3] = 0$.

Proof: Notice that $\{[S_2, \lambda_3], \lambda_1\} \in \mathbf{C} \otimes \mathbf{g}_2$ (the complexification of \mathbf{g}_2). Using the spinor notations: $\{[S_2, \lambda_3], \lambda_1\}_m = (\lambda_1, \Gamma_m \hat{F} S_2^n \Gamma_n \lambda_3)$ where \hat{F} is the Ramond-Ramond 5-form field strength of $AdS_5 \times S^5$ contracted with the Gamma-matrices. Then $\{[S_2, \lambda_3], \lambda_1\} = 0$ would imply that

$$(\lambda_1, X_2^m \Gamma_m \hat{F} S_2^n \Gamma_n \lambda_3) = 0\tag{170}$$

for any vector X_2 . Let us introduce the notation \overline{X}_2 for the vector with the components X_2^m for $m \in \{0, \dots, 4\}$ and $-X_2^m$ for $m \in \{5, \dots, 9\}$. Let \overline{X}_2 run over the space of vectors annihilating λ_3 . Then for such X_2 (170) becomes⁹:

$$\text{Str}(\lambda_1 \lambda_3) \text{Str}(\overline{X}_2 S_2)\tag{171}$$

Therefore our assumption that $\{[S_2, \lambda_3], \lambda_1\} = 0$ implies that the scalar product of S_2 and \overline{X}_2 is zero for any \overline{X}_2 such that $[\overline{X}_2, \lambda_3] = 0$ (*i.e.* for $\overline{X}_2 \in \text{Ann}(\lambda_3)$). Because $\text{Ann}(\lambda_3)$ has the maximal dimension possible of a null-plane, this implies that S_2 itself belongs to the annihilator of λ_3 , *i.e.* $[S_2, \lambda_3] = 0$. This proves the **Lemma**.

┌ If this was not the case, then the action would have a gauge symmetry $w_{1+}^* \mapsto w_{1+}^* + [S_{2+}, \lambda_3]$
└ with S_{2+} satisfying $\{[S_{2+}, \lambda_3], \lambda_1\} = 0$.

We use the notation for the annihilator of a pure spinor:

$$\begin{aligned}\text{Ann}(\lambda_3) &= \text{the subspace of } \mathbf{g}_2 \text{ consisting} \\ &\text{of vectors } X_2 \text{ such that } [X_2, \lambda_3] = 0\end{aligned}\tag{172}$$

Notice that $\text{Ann}(\lambda_3)$ is a $5_{\mathbf{C}}$ -dimensional null-subspace¹⁰ of the complexification of \mathbf{g}_2 . Our Lemma 7.1.1 implies that:

$$\text{Ker}(X_2 \mapsto \{[X_2, \lambda_3], \lambda_1\}) = \text{Ann}(\lambda_3)\tag{173}$$

⁹Notice that $\text{Str}(\overline{X}_2 S_2)$ is the scalar product of \overline{X}_2 and S_2 .

¹⁰A pure spinor defines a null-plane of the maximal possible dimension.

Therefore the image of the map $X_2 \mapsto \{[X_2, \lambda_3], \lambda_1\}$ is also a $5_{\mathbf{C}}$ -dimensional space (because $\mathbf{C} \otimes \mathfrak{g}_2$ is $10_{\mathbf{C}}$ -dimensional). This image is a subspace of $\text{Ann}(\lambda_1)$. But notice that $\text{Ann}(\lambda_1)$ is itself $5_{\mathbf{C}}$ -dimensional. This implies:

$$\text{Im}(X_2 \mapsto \{[X_2, \lambda_3], \lambda_1\}) = \text{Ann}(\lambda_1) \quad (174)$$

Given that $\{\lambda_1, A_1\}$ is in the annihilator of λ_1 , we conclude that there exists such X_2 that $\{\lambda_1, A_1\} + \{\lambda_1, [\lambda_3, X_2]\} = 0$. This proves the existence of the projector \mathcal{P}_{13} . The existence of \mathcal{P}_{31} can be proven similarly.

The Lagrangian depends on the antifields w^* through the last term in (44); therefore the part of the Lagrangian involving w^* is:

$$\mathcal{L}^{AF} = -\text{Str}(w_{1+}^* w_{3-}^*) \quad (175)$$

When we calculate the variation $\Xi_\alpha \mathcal{L}^{AF}$, the projectors \mathcal{P}_{13} and \mathcal{P}_{31} drop out because of the constraint (57). We get:

$$\Xi_\alpha \mathcal{L}^{AF} = -\text{Str}(\partial_+ \alpha g w_{3-}^* g^{-1}) - \text{Str}(g w_{1+}^* g^{-1} \partial_- \alpha) \quad (176)$$

Combining (176) and (166) we get:

$$\Xi_\alpha \mathcal{L} = \frac{1}{4} \text{Str} [j_+ \partial_- \alpha - j_- \partial_+ \alpha] d\tau^+ \wedge d\tau^- = -\frac{1}{4} \text{Str} d\alpha \wedge j \quad (177)$$

Now we observe that Q_1 is an example of Ξ_α for a particular value of α , namely $\alpha = \Lambda_a B^{ab} t_b$:

$$\begin{aligned} Q_1 = 4 \left[B^{ab} \Lambda_a t_b + \partial_+ \Lambda_a B^{ab} (\mathcal{P}_{13}(g t_b g^{-1})_1)^{\dot{\alpha}} \frac{\delta}{\delta w_{1+}^{*\dot{\alpha}}} + \right. \\ \left. + \partial_- \Lambda_a B^{ab} (\mathcal{P}_{31}(g t_b g^{-1})_3)^\alpha \frac{\delta}{\delta w_{3-}^{*\alpha}} \right] \end{aligned} \quad (178)$$

Indeed, Eqs. (177) and (161) imply that so defined Q_1 satisfies:

$$Q_1 \mathcal{L} + Q_0 V_1^{(2)} = 0 \quad (179)$$

which is the defining equation of Q_1 .

In the first term $B^{ab} \Lambda_a t_b$ of Eq. (178) t_b stands for the generators of the $psu(2, 2|4)$ rotations, which act only on the matter fields: $\delta_a g = g t_a$ (they do not touch the ghosts and the antifields).

7.2 Deformation of the BRST current

7.2.1 General procedure for calculating the current density

To calculate the deformed charge density we will use a well-known general procedure. Given the action $S[\phi]$ invariant under some global symmetry transformation $\delta\phi^a = \xi^a$ we consider the position-dependent transformation $\delta_u\phi^a = u(\tau, \sigma)\xi^a$ and calculate the variation of the action. This should be proportional to the derivatives of u and (as any variation) should vanish on-shell:

$$\delta_u S = \int (j_+ \partial_- u - j_- \partial_+ u) \simeq 0 \quad (180)$$

Then it follows that the current j_\pm is conserved: $\partial_+ j_- - \partial_- j_+ = 0$.

7.2.2 Particular case of BRST transformation

Original (undeformed) BRST current Let us start by deriving the BRST charge in the case of pure $AdS_5 \times S^5$. Let us concentrate on Q_L . In this subsection δ will stand for Q_L , and δ_u for Q_L with the replacements:

- $\delta g = \epsilon \lambda_3 g$ replaced with $\delta_u g = \epsilon u \lambda_3 g$
- $\delta w_{1+} = -\epsilon J_{1+} - \epsilon w_{1+}^*$ replaced with $\delta_u w_{1+} = -u \epsilon J_{1+} - u \epsilon w_{1+}^*$
- $\delta w_{3-}^* = D_{0-} \epsilon \lambda_3 - [N_{0-}, \epsilon \lambda_3]$ replaced with $\delta_u w_{3-}^* = u D_{0-} \epsilon \lambda_3 - u [N_{0-}, \epsilon \lambda_3]$

We can tautologically rewrite:

$$\begin{aligned} \delta_u S[g, \lambda, w, w^*] = & (\delta S)[g, u\lambda, u^{-1}w, w^*] + \\ & + (\delta_u S[g, \lambda, w, w^*] - (\delta S)[g, u\lambda, u^{-1}w, w^*]) \end{aligned} \quad (181)$$

We observe that $(\delta S)[g, u\lambda, u^{-1}w, w^*] = 0$ because the action is BRST-invariant. On the other hand, in the second line of (181), the difference between $\delta_u S[g, \lambda, w, w^*]$ and $(\delta S)[g, u\lambda, u^{-1}w, w^*]$ is in two places:

- the variation of the term $(w_{1+} \partial_- \lambda_3)$
- the variation of $-w_{1+}^* w_{3-}^*$

Therefore:

$$\begin{aligned} \epsilon \delta_u S[g, \lambda, w, w^*] - \epsilon (\delta S)[g, u\lambda, u^{-1}w, w^*] = & \int d\tau d\sigma \partial_- u \text{Str} \left(\epsilon (J_{1+} + w_{1+}^*) \lambda_3 + \right. \\ & \left. + w_{1+}^* \epsilon \lambda_3 \right) \end{aligned} \quad (182)$$

The terms containing w_{1+}^* cancel out, and we get:

$$\epsilon j_{L+} = -\text{Str}(J_{1+}\epsilon\lambda_3) \quad (183)$$

$$\epsilon j_{L-} = 0 \quad (184)$$

Notice that the deformation of the BRST transformation given by Eq. (178) does not contribute to the deformation of the BRST current.

Deformation of the BRST current The additional term in the action is:

$$S^{(2)} = \int d\tau d\sigma B^{ab} \left(g^{-1} \left(\frac{dJ_+}{dl} - 4w_{1+}^* \right) g \right)_a \left(g^{-1} \left(\frac{dJ_-}{dl} + 4w_{3-}^* \right) g \right)_b \quad (185)$$

Remember that $Qw_{3-}^* = D_{0-}\lambda_3 - [N_-, \lambda_3]$ and therefore we get:

$$\epsilon \delta_u S^{(2)}[g, \lambda, w, w^*] - \epsilon(\delta S^{(2)})[g, u\lambda, u^{-1}w, w^*] = \quad (186)$$

$$= - \int d\tau d\sigma B^{ab} \left(g^{-1} \left(\frac{dJ_+}{dl} - 4w_{1+}^* \right) g \right)_a (g^{-1}(4\partial_- u\epsilon\lambda_3)g)_b \quad (187)$$

This means that the deformed BRST current is:

$$\epsilon j_{L+} = -\text{Str}(J_{1+}\epsilon\lambda_3) - 4B^{ab}j_{a+}(g^{-1}\epsilon\lambda_3g)_b \quad (188)$$

$$\epsilon j_{L-} = 0 \quad (189)$$

7.2.3 Conservation of the deformed current

Deformed equations of motion Under the variation $\delta_{\xi_3}g = \xi_3g$ we get:

$$\delta_{\xi}J_+ = -D_+(z^{-1}\xi_3) + (z^{-5} - z^{-1})[N_+, \xi_3] + (z^{-4} - 1)[J_{1+}, \xi_3] \quad (190)$$

$$\delta_{\xi}J_- = -D_-(z^{-1}\xi_3) + (z^3 - z^{-1})[N_-, \xi_3] + (z^{-1} - z^3)D_{0-}\xi_3 \quad (191)$$

In particular:

$$\left. \frac{d}{dl} \right|_{l=0} \delta_{\xi}J_+ = D_+\xi_3 - \left[\frac{dJ_+}{dl}, \xi_3 \right] - 4[N_+, \xi_3] - 4[J_{1+}, \xi_3] \quad (192)$$

$$\left. \frac{d}{dl} \right|_{l=0} \delta_{\xi}J_- = D_-\xi_3 - \left[\frac{dJ_-}{dl}, \xi_3 \right] + 4[N_-, \xi_3] - 4D_{0-}\xi_3 \quad (193)$$

The variation of the undeformed action gives:

$$\delta_{\xi}S_0 = \int \text{Str}(\xi_3(D_{0-}J_{1+} - [N_-, J_{1+}] + [J_{1-}, N_+])) \quad (194)$$

The variation of the “small case currents” is:

$$\delta_\xi j_+ = g^{-1} (-4[N_+, \xi_3] - 4[J_{1+}, \xi_3]) g + \partial_+(g^{-1}\xi_3 g) \quad (195)$$

$$\delta_\xi j_- = g^{-1} (4[N_-, \xi_3] + 4[J_{1-} + J_{2-} + J_{3-}, \xi_3]) g - 3\partial_-(g^{-1}\xi_3 g) \quad (196)$$

Therefore:

$$\begin{aligned} \delta_\xi \int d\tau d\sigma B^{ab} j_{a+} j_{b-} = & B^{ab} (g^{-1} (-4[N_+, \xi_3] - 4[J_{1+}, \xi_3]) g)_a j_{b-} + \\ & + B^{ab} j_{a+} (g^{-1} (4[N_-, \xi_3] + 4[J_{1-} + J_{2-} + J_{3-}, \xi_3]) g)_b + \\ & + B^{ab} (\partial_+(g^{-1}\xi_3 g)_a j_{b-} - 3j_{a+} \partial_-(g^{-1}\xi_3 g)_b) \end{aligned} \quad (197)$$

The last term is equivalent to $\int 4(\partial_- j_{a+}) B^{ab} (g^{-1}\xi_3 g)_b$. We conclude that the deformed equation of motion for J_{1+} is:

$$\begin{aligned} & D_{0-} J_{1+} - [N_-, J_{1+}] + [J_{1-}, N_+] + \\ & + 4[N_+ + J_{1+}, gt_a g^{-1}]_1 B^{ab} j_{b-} - 4j_{a+} B^{ab} [N_- + J_{1-} + J_{2-} + J_{3-}, gt_b g^{-1}]_1 + \\ & + 4B^{ab} (\partial_- j_{a+}) (gt_b g^{-1})_1 = 0 \end{aligned} \quad (198)$$

We will also need the deformed equations of motion for λ_3 , which is obtained by varying the action with respect to w_{1+} :

$$D_{0-} \lambda_3 - [N_-, \lambda_3] - 4B^{ab} [(gt_a g^{-1})_0, \lambda_3] j_{b-} = 0 \quad (199)$$

Holomorphicity of the current Consider the derivative of the current given by Eq. (188):

$$\begin{aligned} -\partial_- j_{L+} = & \text{Str} (J_{1+} (D_{0-} \epsilon \lambda_3 - [N_-, \epsilon \lambda_3]) + (D_{0-} J_{1+} - [N_-, J_{1+}]) \epsilon \lambda_3) + \\ & + \partial_- (4B^{ab} j_{a+} (g^{-1} \epsilon \lambda_3 g)_b) \end{aligned} \quad (200)$$

Substitution of (198) and (199) into this formula gives $\partial_- j_{L+} = 0$.

7.3 Relation between $W_2^{(0)}$ and the Schouten bracket on $\Lambda^\bullet \mathfrak{g}$

Equation (139) tells us that a necessary condition for the deformed theory to be BRST invariant to the order ε^2 is that $W_2^{(0)}$ is BRST exact. Here we will explicitly calculate $W_2^{(0)}$ for the beta-deformation and express it in terms of the Schouten bracket on $\Lambda^2 \mathfrak{g}$.

We start with the observation that actually:

$$Q_0 V_1^{(2)} + Q_1 \mathcal{L}^{(2)} = 0 \quad (201)$$

In other words $X_1^{(1)} = 0$.

Notice that generally speaking we only have Eq. (132), but in our particular case $V = \frac{1}{2}B^{ab}j_a \wedge j_b$ we claim a stronger Eq. (201).

It remains to calculate $I_{Q_1^2}$. Let us first calculate Q_1^2 . Let us split $Q_1 = Q_1^{\mathbf{F}} + Q_1^{\mathbf{AF}}$ where $Q_1^{\mathbf{F}} = 4\Lambda_a B^{ab}t_b$ is the first term on the right hand side of (178) and $Q_1^{\mathbf{AF}}$ is the sum of the remaining two terms (*i.e.* \mathbf{F} stands for fields and \mathbf{AF} for antifields). We get:

$$\begin{aligned} [Q_1^{\mathbf{F}}(\epsilon), Q_1^{\mathbf{F}}(\epsilon')] &= 16\Lambda_a(\epsilon)B^{ap}\Lambda_b(\epsilon')B^{bq}f_{pq}{}^r t_r + \\ &+ 16 \times 2\Lambda_a(\epsilon)B^{ap}f_{pb}{}^r \Lambda_r(\epsilon')B^{bq}t_q = \\ &= 16 \times 3 B^{p[a}f_{pq}{}^b B^{c]q}\Lambda_a(\epsilon)\Lambda_b(\epsilon')t_c \end{aligned} \quad (202)$$

This has similar structure to $Q_1^{\mathbf{F}}$; namely it is a $psu(2, 2|4)$ -rotation, but not a global symmetry because the parameter of the rotation is space-time dependent. We have $[Q_1^{\mathbf{AF}}(\epsilon), Q_1^{\mathbf{AF}}(\epsilon')] = 0$, and $[Q_1^{\mathbf{F}}(\epsilon), Q_1^{\mathbf{AF}}(\epsilon')]$ is given by:

$$\begin{aligned} [Q_1^{\mathbf{F}}(\epsilon), Q_1^{\mathbf{AF}}(\epsilon')]w_{1+}^* &= 16 B^{cd}B^{eb}f_{de}{}^a \partial_+(\Lambda_c(\epsilon)\Lambda_a(\epsilon')) \mathcal{P}_{13}(gt_b g^{-1})_1 - \\ &- 16 B^{cd}B^{eb}f_{de}{}^a \Lambda_c(\epsilon)\partial_+\Lambda_b(\epsilon') \mathcal{P}_{13}(gt_a g^{-1})_1 = \\ &= 16 B^{cd}f_{de}{}^a B^{eb}\partial_+\Lambda_c(\epsilon)\Lambda_a(\epsilon') \mathcal{P}_{13}(gt_b g^{-1})_1 - \\ &- 16 B^{cd}f_{de}{}^a B^{eb}\partial_+\Lambda_a(\epsilon)\Lambda_c(\epsilon') \mathcal{P}_{13}(gt_b g^{-1})_1 + \\ &+ 16 B^{cd}f_{de}{}^a B^{eb}\partial_+\Lambda_b(\epsilon)\Lambda_c(\epsilon') \mathcal{P}_{13}(gt_a g^{-1})_1 = \\ &= 3 \times 16 B^{d[c}f_{de}{}^a B^{b]e}\partial_+\Lambda_c(\epsilon)\Lambda_a(\epsilon') \mathcal{P}_{13}(gt_b g^{-1})_1 \end{aligned} \quad (203)$$

This implies:

$$I_{Q_1^2}^{(1)} = 3 \times 16 B^{p[a}f_{pq}{}^b B^{c]q}\Lambda_a(\epsilon)\Lambda_b(\epsilon')j_c \quad (204)$$

and therefore:

$$W_2^{(0)} = 3 \times 16 B^{pa}f_{pq}{}^b B^{cq}\Lambda_a\Lambda_b\Lambda_c \quad (205)$$

This means that $W_2^{(0)}$ is expressed in terms of the Schouten bracket on $\Lambda^\bullet \mathfrak{g}$:

$$[[B_1, B_2]]^{abc} = B_1^{[a|e|}f_{ef}{}^b B_2^{f|c]} \quad (206)$$

7.4 Could a nonzero $[[B, B]]$ be harmless?

7.4.1 Operator $W_2^{(0)}$ may be Q -exact

We have seen that the obstacle to extending the deformation to the second order in ε is $[[B, B]]^{abc}\Lambda_a\Lambda_b\Lambda_c$. But a nonzero $[[B, B]]$ does not yet mean that the deformation is obstructed, because $[[B, B]]^{abc}\Lambda_a\Lambda_b\Lambda_c$ can still be Q_{BRST} -exact:

$$[[B, B]]^{abc}\Lambda_a\Lambda_b\Lambda_c \stackrel{?}{=} Q_0 T \quad (207)$$

Example of a Q -exact expression of the ghost number 3 Let us consider the following operator of the ghost number 2:

$$T = A^{ma} \bar{\Lambda}_m \Lambda_a \quad (208)$$

where A^{ma} is some tensor which does not need to have any special symmetry properties under the exchange $m \leftrightarrow a$. In this case we get:

$$Q_0 T = -A^{ma} f_m^{bc} \Lambda_b \Lambda_c \Lambda_a \quad (209)$$

Therefore, if:

$$[[B, B]]^{abc} = A^{m[a} f_m^{bc]} \quad (210)$$

then such $[[B, B]]$ is harmless. In particular, such harmless $[[B, B]]$ arise in the following situation. Consider $[[B_1, B_2]]^{abc} \Lambda_a \Lambda_b \Lambda_c$ in the special case when $B_1^{ab} = G^l f_l^{ab}$. We get:

$$G^l f_l^{e[a} f_{ef}^b B_2^{c]f} = -G^l [t^a, [t^b, t_f]]_l B_2^{c]f} = \frac{1}{2} f_{fl}^g G^l f^{[ab}_g B_2^{c]f} \quad (211)$$

This expression is proportional to f_g^{ab} . Therefore in this case $[[B_1, B_2]]^{abc} \Lambda_a \Lambda_b \Lambda_c$ is BRST-exact:

$$[[B_1, B_2]]^{abc} \Lambda_a \Lambda_b \Lambda_c = Q(f_{mn}^a G^m B_2^{nb} \bar{\Lambda}_a \Lambda_b) \quad (212)$$

This implies that the condition that $[[B, B]]$ is exact is correctly defined on the equivalence classes of $B^{ab} \sim B^{ab} + f^{ab}_c G^c$ in agreement with Sections 3.1 and 5.2. Comparing (212) with (210) we see that in this case $A^{ma} = f_{pq}^m G^p B^{qa}$. Notice that this A^{ma} is not antisymmetric in $a \leftrightarrow m$; but the *antisymmetrization* of A is a Schouten bracket $[[G, B]]$.

7.4.2 But $W^{(0)}$ is of ghost number 3; isn't it always Q_0 -exact?

There is no nontrivial BRST cohomology in the ghost number 3, therefore strictly speaking $W_2^{(0)}$ is always Q_0 -exact. Since $Q_0 W_2^{(0)} = 0$ we should always be able to find T such that $W_2^{(0)} = Q_0 T$. However, this is not true if we also impose some additional constraints on T . There are two possible constraints on T :

1. Covariance, *i.e.* we demand T to transform covariantly under $psu(2, 2|4)$. Apriori it only transforms covariantly modulo $\text{Ker } Q_0$; see [3].
2. Absence of resonant terms; in other words T is periodic in the global time of AdS_5 . Notice that this would be automatically satisfied if we impose the covariance.

Considerations similar to [3] show that there are the following obstructions to the covariance of T :

$$\begin{aligned} & H^1 \left(\mathfrak{g}, \text{Hom}_{\mathbf{C}} \left(\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}, \begin{bmatrix} \text{physical} \\ \text{states} \end{bmatrix} \right) \right) \\ & H^2 \left(\mathfrak{g}, \text{Hom}_{\mathbf{C}} \left(\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}, \begin{bmatrix} \text{conserved} \\ \text{charges} \end{bmatrix} \right) \right) \\ & H^3 (\mathfrak{g}, \text{Hom}_{\mathbf{C}} (\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}, \mathbf{C})) \end{aligned} \quad (213)$$

7.4.3 Open problem: how to tell if $W_2^{(0)}$ is BRST-exact in the covariant sub-complex?

We do not know the answer to this question.

It appears to us that our example (210) does not exhaust all the possibilities for $[[B, B]]$ to be Q_0 -exact. Indeed we prove in Appendix C that $[[B, B]]$ necessarily has some odd indices. But there are examples of solutions (complex β -deformations) where $[[B, B]]$ only has even indices.

The study of the obstructions (213) is one possible approach, but we will not do it in this paper. In this Section we will give a guess about what the image of Q_0 might be.

Consider the condition (207):

$$[[B, B]]^{abc} \Lambda_a \Lambda_b \Lambda_c \stackrel{?}{=} Q_0 T_2^{(0)} \quad (214)$$

with the restriction that $T_2^{(0)}$ transforms covariantly under the $\mathfrak{psu}(2, 2|4)$ -rotation of g and B . More precisely:

$$T_2^{(0)} \in \text{ind}_{\mathfrak{g}_0}^{\mathfrak{g}} (\mathfrak{g}_3 \bullet \mathfrak{g}_3 \oplus \mathfrak{g}_3 \otimes \mathfrak{g}_1 \oplus \mathfrak{g}_1 \bullet \mathfrak{g}_1) \quad (215)$$

Here \bullet means the graded-symmetric product (because \mathfrak{g}_1 and \mathfrak{g}_3 are odd subspaces, this is actually the antisymmetric product in the usual sense). Moreover we are interested in a *linear subspace* of (215) consisting of the $T_2^{(0)}$ such that $QT_2^{(0)}$ belongs to the linear space generated by the expressions of the form:

$$\Lambda_a \Lambda_b \Lambda_c \quad (216)$$

Let \mathcal{T} denotes the subspace of such $T_2^{(0)}$:

$$\mathcal{T} \subset \text{ind}_{\mathfrak{g}_0}^{\mathfrak{g}} (\mathfrak{g}_3 \bullet \mathfrak{g}_3 \oplus \mathfrak{g}_3 \otimes \mathfrak{g}_1 \oplus \mathfrak{g}_1 \bullet \mathfrak{g}_1) \quad (217)$$

$$Q\mathcal{T} \subset \text{expressions of the form } W^{abc} \Lambda_a \Lambda_b \Lambda_c \text{ where } W \in \Lambda^3 \mathfrak{g} \quad (218)$$

We do not have a complete description of \mathcal{T} . We want to point out the following:

1. expressions of the form (208) are in \mathcal{T}
2. but we think¹¹ that \mathcal{T} is not exhausted by the expressions of the type (208)
3. the image of Q is a linear subspace, therefore the condition on B following from (214) should be of the type:

$$[[B, B]] \text{ belongs to a certain subspace } S \subset \Lambda^3 \mathfrak{g} \quad (219)$$

Let us call S the “harmless subspace”.

Let us look at some invariant subspaces in $\Lambda^3 \mathfrak{g}$. Consider $\widehat{\mathfrak{g}}' = u(2, 2|4)$. The adjoint representation of $\widehat{\mathfrak{g}}'$ is a tensor product of the fundamental and antifundamental representations; therefore $W \in \Lambda^3 \widehat{\mathfrak{g}}'$ is represented as a matrix W_{jln}^{ikm} , graded-antisymmetric with respect to the

¹¹because of the existence of complex β -deformations, see Section 9

exchange of pairs $\begin{pmatrix} i \\ j \end{pmatrix} \leftrightarrow \begin{pmatrix} k \\ l \end{pmatrix} \leftrightarrow \begin{pmatrix} m \\ n \end{pmatrix}$. Let us introduce the transposition operators $(12)_{up}$, $(13)_{up}$, $(23)_{up}$, $(12)_{dn}$, $(13)_{dn}$ and $(23)_{dn}$ in the tensor product $\widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}'$, for example:

$$(12)_{up} T_{jln}^{ikm} = T_{jln}^{kim} \quad , \quad (13)_{up} T_{jln}^{ikm} = T_{jln}^{mki} \quad , \quad (23)_{dn} T_{jln}^{ikm} = T_{jnl}^{ikm} \quad (220)$$

For example, $\Lambda^3 \widehat{\mathbf{g}}'$ is the image of $\widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}'$ under the following pair-wise-antisymmetrization operator:

$$1 - (12)_{up}(12)_{dn} - (13)_{up}(13)_{dn} - (23)_{up}(23)_{dn} + (12)_{up}(23)_{up}(12)_{dn}(23)_{dn} + (23)_{up}(12)_{up}(23)_{dn}(12)_{dn}$$

Let us denote this operator \mathcal{A} (pairwise antisymmetrization). Then we denote:

$$L_{(2,0)} = \mathcal{A}(1 + (23)_{up})(1 + (12)_{up}) \quad \widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}' \quad (221)$$

$$L_{(0,2)} = \mathcal{A}(1 + (12)_{dn})(1 + (23)_{dn}) \quad \widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}' \otimes \widehat{\mathbf{g}}' \quad (222)$$

Consider the complex structure \mathcal{I} which we introduced in Section 5.1.3. Consider the expression $\llbracket e^{\phi \mathcal{I}} B, e^{\phi \mathcal{I}} B \rrbracket$. As a function of ϕ it contains a part proportional to $e^{2i\phi}$, a part proportional to $e^{-2i\phi}$ and a constant part. The part proportional to $e^{2i\phi}$ is in $L_{(2,0)}$, and the part proportional to $e^{-2i\phi}$ is in $L_{(0,2)}$. Therefore this condition:

$$\llbracket B, B \rrbracket \in L_{(2,0)} + L_{(0,2)} + \mathbf{1} \wedge \mathbf{g} \wedge \mathbf{g} \quad (223)$$

is invariant under $B \rightarrow e^{\phi \mathcal{I}} B$. Also, expressions of the type (209) are in $L_{(2,0)} + L_{(0,2)} + \mathbf{1} \wedge \mathbf{g} \wedge \mathbf{g}$. Indeed, let us take $X_{jln}^{ikm} = \delta_j^k A_{ln}^{im}$. Then $\mathcal{A}X$ is identified with (209). On the other hand we observe:

$$([1 - (23)_{up}(12)_{up}]X)_{jln}^{ikm} = X_{jln}^{ikm} - ((12)_{up}X)_{jln}^{imk} = X_{jln}^{ikm} - X_{jln}^{mik} = X_{jln}^{ikm} - \delta_j^i A_{ln}^{mk} \quad (224)$$

Taking into account that:

$$1 = \frac{1}{2}(1 - (23)_{up}(12)_{up}) + \frac{1}{4}(1 + (23)_{up})(1 + (12)_{up}) + \frac{1}{4}(1 - (23)_{up})(1 - (12)_{up}) \quad (225)$$

this implies that $\mathcal{A}X$ is in $L_{(2,0)} + L_{(0,2)} + \mathbf{1} \wedge \mathbf{g} \wedge \mathbf{g}$.

It is natural to conjecture that (223) is sufficient for (214). But this is only a guess.

7.5 Calculation of $V_2^{(2)}$

We will start with calculating $Q_1 j_{a+}$. We get:

$$\begin{aligned} Q_1 j_{a+} &= 4 \Lambda_b B^{bc} f_{ca}^e j_{e+} + \\ &+ 4 \partial_+ \Lambda_b B^{bc} \text{Str} \left(t_a g^{-1} \left((gt_c g^{-1})_3 + 2(gt_c g^{-1})_2 + 3(gt_c g^{-1})_1 - \right. \right. \\ &\quad \left. \left. - 4\mathcal{P}_{13}(gt_c g^{-1})_1 \right) g \right) \end{aligned} \quad (226)$$

$$\begin{aligned} Q_1 j_{a-} &= 4 \Lambda_b B^{bc} f_{ca}^e j_{e-} - \\ &- 4 \partial_- \Lambda_b B^{bc} \text{Str} \left(t_a g^{-1} \left(3(gt_c g^{-1})_3 + 2(gt_c g^{-1})_2 + (gt_c g^{-1})_1 - \right. \right. \\ &\quad \left. \left. - 4\mathcal{P}_{31}(gt_c g^{-1})_3 \right) g \right) \end{aligned} \quad (227)$$

This means:

$$\begin{aligned}
Q_1 B^{ab} j_{a+} j_{b-} &= 4 B^{cd} B^{ab} \Lambda_c (f_{da}{}^e j_{e+} j_{b-} + f_{db}{}^e j_{a+} j_{e-}) + \\
&+ 4 B^{cd} B^{ab} \partial_+ \Lambda_c \text{Str} (gt_a g^{-1} ((gt_d g^{-1})_3 + 2(gt_d g^{-1})_2 + 3(gt_d g^{-1})_1 - \\
&\quad - 4 \mathcal{P}_{13}(gt_d g^{-1})_1)) j_{b-} + \\
&+ 4 B^{cd} B^{ab} j_{a+} \partial_- \Lambda_c \text{Str} (gt_b g^{-1} (-3(gt_d g^{-1})_3 - 2(gt_d g^{-1})_2 - (gt_d g^{-1})_1 + \\
&\quad + 4 \mathcal{P}_{31}(gt_d g^{-1})_3)) = \\
&= 4 B^{cd} B^{ab} \Lambda_c (f_{da}{}^e j_{e+} j_{b-} + f_{db}{}^e j_{a+} j_{e-}) + \\
&+ 4 B^{cd} B^{ab} Q_0(j_{+c} j_{b-}) \text{Str} (gt_a g^{-1} ((gt_d g^{-1})_3 + 2(gt_d g^{-1})_2 + 3(gt_d g^{-1})_1 - \\
&\quad - 4 \mathcal{P}_{13}(gt_d g^{-1})_1)) \tag{228}
\end{aligned}$$

Now we are going to use the condition (210). Let us first assume that $A = 0$, and then consider the case when A is nonzero.

7.5.1 The case when $\llbracket B, B \rrbracket = 0$

Taken into account that $\llbracket B, B \rrbracket = 0$ we can transform in the first line of (228):

$$B^{cd} B^{ab} \Lambda_c (f_{da}{}^e j_{e+} j_{b-} + f_{db}{}^e j_{a+} j_{e-}) = -B^{dc} B^{ab} \Lambda_e f_{da}{}^e j_{c+} j_{b-} \tag{229}$$

Observe that:

$$\begin{aligned}
Q_{0L} \text{Str} (gt_a g^{-1} ((gt_d g^{-1})_3 + 2(gt_d g^{-1})_2 + 3(gt_d g^{-1})_1 - 4\mathcal{P}_{13}(gt_d g^{-1})_1)) &= \\
= \text{Str} ([\lambda_3, gt_a g^{-1}] ((gt_d g^{-1})_3 + 2(gt_d g^{-1})_2 + 3(gt_d g^{-1})_1 - 4\mathcal{P}_{13}(gt_d g^{-1})_1)) &+ \\
+ \text{Str} (gt_a g^{-1} ([\lambda_3, (gt_d g^{-1})_0] + 2[\lambda_3, (gt_d g^{-1})_3] + 3[\lambda_3, (gt_d g^{-1})_2])) & \tag{230}
\end{aligned}$$

Notice that $\text{Str} ([\lambda_3, gt_a g^{-1}] \mathcal{P}_{13}(gt_d g^{-1})_1) = \text{Str} ([\lambda_3, gt_a g^{-1}] (gt_d g^{-1})_1)$ and therefore:

$$\begin{aligned}
Q_{0L} \text{Str} (gt_a g^{-1} ((gt_d g^{-1})_3 + 2(gt_d g^{-1})_2 + 3(gt_d g^{-1})_1 - 4\mathcal{P}_{13}(gt_d g^{-1})_1)) &= \\
= \text{Str} ([\lambda_3, gt_a g^{-1}] ((gt_d g^{-1})_3 + 2(gt_d g^{-1})_2 - (gt_d g^{-1})_1)) &+ \\
+ \text{Str} (gt_a g^{-1} ([\lambda_3, (gt_d g^{-1})_0] + 2[\lambda_3, (gt_d g^{-1})_3] + 3[\lambda_3, (gt_d g^{-1})_2])) & \tag{231}
\end{aligned}$$

Now we split this up:

$$\begin{aligned}
&\text{Str} ([\lambda_3, (gt_a g^{-1})_2] (gt_d g^{-1})_3 + 2[\lambda_3, (gt_a g^{-1})_3] (gt_d g^{-1})_2 - [\lambda_3, (gt_a g^{-1})_0] (gt_d g^{-1})_1 + \\
&+ (gt_a g^{-1})_1 [\lambda_3, (gt_d g^{-1})_0] + 2(gt_a g^{-1})_2 [\lambda_3, (gt_d g^{-1})_3] + 3(gt_a g^{-1})_3 [\lambda_3, (gt_d g^{-1})_2]) = \\
= &\text{Str} (-[\lambda_3, (gt_a g^{-1})_2] (gt_d g^{-1})_3 - [\lambda_3, (gt_a g^{-1})_3] (gt_d g^{-1})_2 - \\
&- [\lambda_3, (gt_a g^{-1})_1] (gt_d g^{-1})_0 - [\lambda_3, (gt_a g^{-1})_0] (gt_d g^{-1})_1) \tag{232}
\end{aligned}$$

Finally, this can be written as:

$$\begin{aligned} Q_{0L} \text{Str} (gt_ag^{-1} ((gt_dg^{-1})_3 + 2(gt_dg^{-1})_2 + 3(gt_dg^{-1})_1 - 4\mathcal{P}_{13}(gt_dg^{-1})_1)) = \\ = -\text{Str}(\lambda_3[gt_ag^{-1}, gt_dg^{-1}]) = -f_{ad}^e(g^{-1}\lambda_3g)_e \end{aligned} \quad (233)$$

Similarly, let us calculate the action of Q_{0R} on the same expression:

$$\begin{aligned} Q_{0R} \text{Str} (gt_ag^{-1} ((gt_dg^{-1})_3 + 2(gt_dg^{-1})_2 + 3(gt_dg^{-1})_1 - 4\mathcal{P}_{13}(gt_dg^{-1})_1)) = \\ = \text{Str} ([\lambda_1, gt_ag^{-1}] ((gt_dg^{-1})_3 + 2(gt_dg^{-1})_2 + 3(gt_dg^{-1})_1 - 4\mathcal{P}_{13}(gt_dg^{-1})_1)) + \\ + \text{Str} (gt_ag^{-1} ([\lambda_1, (gt_dg^{-1})_2] + 2[\lambda_1, (gt_dg^{-1})_1] - [\lambda_1, (gt_dg^{-1})_0])) \end{aligned} \quad (234)$$

where we have used that $\mathcal{P}_{13}[\lambda_1, x_0] = [\lambda_1, x_0]$ for any x . We can write this in the following form:

$$\begin{aligned} \text{Str} ([\lambda_1, (gt_ag^{-1})_0](gt_dg^{-1})_3 + 2[\lambda_1, (gt_ag^{-1})_1](gt_dg^{-1})_2 + 3[\lambda_1, (gt_ag^{-1})_2](gt_dg^{-1})_1) + \\ + \text{Str} ((gt_ag^{-1})_1[\lambda_1, (gt_dg^{-1})_2] + 2(gt_ag^{-1})_2[\lambda_1, (gt_dg^{-1})_1] - (gt_ag^{-1})_3[\lambda_1, (gt_dg^{-1})_0]) = \\ = \text{Str} ([\lambda_1, (gt_ag^{-1})_0](gt_dg^{-1})_3 + [\lambda_1, (gt_ag^{-1})_3](gt_dg^{-1})_0) + \\ + \text{Str} ([\lambda_1, (gt_ag^{-1})_1](gt_dg^{-1})_2 + [\lambda_1, (gt_ag^{-1})_2](gt_dg^{-1})_1) \end{aligned}$$

Therefore:

$$\begin{aligned} Q_{0R} \text{Str} (gt_ag^{-1} ((gt_dg^{-1})_3 + 2(gt_dg^{-1})_2 + 3(gt_dg^{-1})_1 - 4\mathcal{P}_{13}(gt_dg^{-1})_1)) = \\ = \text{Str} (g^{-1}\lambda_1g[t_a, t_d]) = f_{ad}^e(g^{-1}\lambda_1g)_e \end{aligned} \quad (235)$$

Combining (233) and (235) with (228) and (229) we get:

$$Q_1V_1^{(2)} = -Q_0V_2^{(2)} \quad (236)$$

where

$$\begin{aligned} V_2^{(2)} = -4 B^{cd}B^{ab} j_{c+}j_{b-} \text{Str} ((gt_ag^{-1})_1(gt_dg^{-1})_3 + 2(gt_ag^{-1})_2(gt_dg^{-1})_2 + \\ + 3(gt_ag^{-1})_3(gt_dg^{-1})_1 - 4(gt_ag^{-1})_3\mathcal{P}_{13}(gt_dg^{-1})_1) \end{aligned} \quad (237)$$

Notice that $V_2^{(2)}$ is non-polynomial in pure spinors because of \mathcal{P}_{13} . But we will see that the nonlocality actually cancels out in the classical action if we substitute the classical values for the antifields w^* (which are non-zero after the deformation). In other words, we remove the terms with \mathcal{P} by a shift of w^* .

7.5.2 The case when $\llbracket B, B \rrbracket$ is of the form (210)

Now we have to explain what happens when $\llbracket B, B \rrbracket$ is nonzero, but is Q -exact in the sense of (210). Then (229) fails and consequently instead of (236) we are getting this:

$$Q_1 V_1^{(2)} = -Q_0 V_2^{(2)} + 3A^{m[a} f_m{}^{bc]} \Lambda_a j_{b+} j_{c-} \quad (238)$$

The second term on the right hand side can be transformed as follows:

$$\begin{aligned} & A^{ma} f_m{}^{bc} (\Lambda_a j_b \wedge j_c + \Lambda_b j_c \wedge j_a + \Lambda_c j_a \wedge j_b) \simeq \\ & \simeq -A^{ma} \Lambda_a d(g^{-1} \frac{d^2 J}{dl^2} g)_m + 2A^{ma} [\Lambda, j]_m \wedge j_a = \\ & = -d \left(A^{ma} \Lambda_a (g^{-1} \frac{d^2 J}{dl^2} g)_m \right) + Q_0 \left(A^{ma} j_a (g^{-1} \frac{d^2 J}{dl^2} g)_m \right) - A^{ma} j_a \wedge Q_0 (g^{-1} \frac{d^2 J}{dl^2} g)_m + \\ & \quad + 2A^{ma} [\Lambda, j]_m \wedge j_a \end{aligned} \quad (239)$$

Here we can use:

$$\begin{aligned} \epsilon Q_0 (g^{-1} \frac{d^2 J}{dl^2} g) &= g^{-1} \left[\frac{d^2 J}{dl^2}, \epsilon \lambda \right] g - g^{-1} \frac{d^2}{dl^2} (D \epsilon \lambda) g = \\ &= -2 \left[g^{-1} \frac{dJ}{dl} g, g^{-1} \frac{d\epsilon \lambda}{dl} g \right] - g^{-1} D(\epsilon \lambda) g = \\ &= -2[\epsilon \Lambda, j] - d(\epsilon \bar{\Lambda}) \end{aligned} \quad (240)$$

and finally obtain:

$$\begin{aligned} & A^{ma} f_m{}^{bc} (\Lambda_a j_b \wedge j_c + \Lambda_b j_c \wedge j_a + \Lambda_c j_a \wedge j_b) \simeq \\ & \simeq -d \left(A^{ma} \Lambda_a (g^{-1} \frac{d^2 J}{dl^2} g)_m + A^{ma} \bar{\Lambda}_m j_a \right) + Q_0 \left(A^{ma} j_a (g^{-1} \frac{d^2 J}{dl^2} g)_m \right) \end{aligned} \quad (241)$$

Therefore with $A \neq 0$ we get:

$$Q_1 V_1^{(2)} \simeq -Q_0 \left(V_2^{(2)} + A^{ma} j_a (g^{-1} \frac{d^2 J}{dl^2} g)_m \right) + d(\text{smth}) \quad (242)$$

7.6 Taking into account nonzero classical values of the antifields

7.6.1 Second order correction to the deformed action

The terms in $\mathcal{L} + V_1^{(2)}$ containing the antifields are the following:

$$- \text{Str}(w_{1+}^* w_{3-}^*) - 4(g^{-1} w_{1+}^* g)_a B^{ab} j_{b-} + 4j_{a+} B^{ab} (g^{-1} w_{3-}^* g)_b \quad (243)$$

This means that the classical values of the antifields are:

$$w_{3-}^*|_{cl} = -4\mathcal{P}_{31}(gt_ag^{-1})_3 B^{ab}j_{b-} \quad (244)$$

$$w_{1+}^*|_{cl} = 4\mathcal{P}_{13}j_{a+} B^{ab}(gt_bg^{-1})_1 \quad (245)$$

When we substitute these classical values back into the action, we get:

$$w_{1+}^*|_{cl} w_{3-}^*|_{cl} = -16j_{c+}j_{b-} B^{cd}B^{ab}\text{Str}\left((gt_dg^{-1})_1\mathcal{P}_{31}(gt_ag^{-1})_3\right) \quad (246)$$

Combining this with (237) we get:

$$\begin{aligned} V_2^{(2)} + w_{1+}^*|_{cl}w_{3-}^*|_{cl} = & -4 B^{cd}B^{ab} j_{c+}j_{b-} \text{Str}\left((gt_ag^{-1})_1(gt_dg^{-1})_3 + \right. \\ & + 2(gt_ag^{-1})_2(gt_dg^{-1})_2 + \\ & \left. + 3(gt_ag^{-1})_3(gt_dg^{-1})_1\right) \end{aligned} \quad (247)$$

This formula describes the second order deformation of the classical action.

7.6.2 BRST transformation of the shifted antifields

Taking into account (244) and (245) we define the shifted antifields:

$$\underline{w}_{3-}^* = w_{3-}^* - w_{3-}^*|_{cl} = w_{3-}^* + 4\mathcal{P}_{31}(gt_ag^{-1})_3 B^{ab}j_{b-} \quad (248)$$

$$\underline{w}_{1+}^* = w_{1+}^* - w_{1+}^*|_{cl} = w_{1+}^* - 4\mathcal{P}_{13}j_{a+} B^{ab}(gt_bg^{-1})_1 \quad (249)$$

In terms of these shifted antifields the BRST transformation $Q_0 + \varepsilon Q_1$ (where Q_1 is given by (178)) is:

$$\begin{aligned} (Q_0 + \varepsilon Q_1)\underline{w}_{1+}^* &= D_{0+}\lambda_1 - [N_+, \lambda_1] + 4\varepsilon j_{a+}B^{ab}[(gt_bg^{-1})_0, \lambda_1] \\ (Q_0 + \varepsilon Q_1)\underline{w}_{3-}^* &= D_{0-}\lambda_3 - [N_-, \lambda_3] + 4\varepsilon j_{a-}B^{ab}[(gt_bg^{-1})_0, \lambda_3] \end{aligned} \quad (250)$$

7.7 Conclusion

The action at the second order is given by:

$$\begin{aligned} S = \frac{R^2}{\pi} \int d^2z \text{Str} & \left(\frac{1}{2}J_{2+}J_{2-} + \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-} + \right. \\ & + w_{1+}\partial_-\lambda_3 + w_{3-}\partial_+\lambda_1 + N_{0+}J_{0-} + N_{0-}J_{0+} - N_{0+}N_{0-} + \\ & \left. + \frac{1}{2}\varepsilon B^{ab}j_{[a} \wedge j_{b]} - \right. \end{aligned} \quad (251)$$

$$\begin{aligned} & - 4\varepsilon^2 B^{cd}B^{ab} j_{c+}j_{b-} \text{Str}\left((gt_ag^{-1})_1(gt_dg^{-1})_3 + \right. \\ & \quad + 2(gt_ag^{-1})_2(gt_dg^{-1})_2 + \\ & \quad \left. + 3(gt_ag^{-1})_3(gt_dg^{-1})_1\right) - \\ & \left. - \underline{w}_{1+}^*\underline{w}_{3-}^* \right) \end{aligned} \quad (252)$$

7.8 Comments

7.8.1 About higher orders

We have started with $V_1^{(2)} = B^{ab}j_{a+}j_{b-}$ and obtained $V_2^{(2)} = -4B^{ap}M_{pq}B^{qb}j_{c+}j_{b-}$ where

$$M_{pq} = \text{Str} \left((gt_ag^{-1})_1(gt_dg^{-1})_3 + 2(gt_ag^{-1})_2(gt_dg^{-1})_2 + 3(gt_ag^{-1})_3(gt_dg^{-1})_1 \right) \quad (253)$$

Notice that while $V_1^{(2)}$ is parity-odd $V_2^{(2)}$ is not. This corresponds to the fact that there is a nonzero deformation of the metric at the second order. Also, notice the schematic pattern in going from the first order vertex to the second order vertex:

$$B^{ab} \longrightarrow B^{ap}M_{pq}B^{qb} \quad (254)$$

We conjecture that higher orders follow the same pattern.

7.8.2 About the gauge transformation $B^{ab} \mapsto B^{ab} + f^{ab}_c G^c$

As we explained in Section 3.2 the gauge transformation

$$B^{ab} \mapsto B^{ab} + f^{ab}_c G^c \quad (255)$$

should be accompanied by a field redefinition $G^a \mathcal{X}_a$. Therefore the condition of the gauge invariance at the second order in ϵ is:

$$f^{ab}_c G^c \frac{\delta}{\delta B^{ab}} V_2^{(2)} + G^a \mathcal{X}_a V_1^{(2)} = d(\text{smth}) \quad (256)$$

This means that $V_2^{(2)}$ is not invariant under the gauge transformation (255) in the naive sense, but rather in the sense of Eq. (256).

8 Properties of the Schouten bracket on $\Lambda^\bullet \mathfrak{g}$

8.1 Projection to $\mathfrak{g} \otimes \mathfrak{g}$

Given $a \wedge b \wedge c \in \Lambda^3 \mathfrak{g}$ we consider:

$$[a \wedge b \wedge c] = [a, b] \otimes c - [a, c] \otimes b + [b, c] \otimes a \in \mathfrak{g} \otimes \mathfrak{g} \quad (257)$$

If the internal commutator of B (defined in Section 4) vanishes, then:

$$[[B, B]] \in \mathfrak{g} \bullet \mathfrak{g} \quad (258)$$

where \bullet means the symmetric product. More precisely, this is $f^a_{kl} f^b_{mn} B^{km} B^{ln}$.

8.2 From the r -matrix point of view

Suppose that B satisfies $\llbracket B, B \rrbracket = 0$. Then we can think of B as a classical r -matrix. It defines the Poisson bracket on \mathfrak{g} , and therefore the structure of the Lie algebra on \mathfrak{g}^* , in the following way [24]:

$$[X, Y]_B^{(1)} = \iota(B) d(X \wedge Y) = \text{ad}_{X_b B^{bc} t_c}^* Y - (X \leftrightarrow Y) \quad (259)$$

In this formula $d(X \wedge Y)$ is the differential on $\Lambda^\bullet \mathfrak{g}^*$, which is the same d as defines the Lie algebra cohomology. This differential is “dual” to the Lie bracket on \mathfrak{g} , in the following sense. Remember that in our notations the coordinates of an element $\xi \in \mathfrak{g}$ are enumerated with the upper indices:

$$\xi = \xi^a t_a \in \mathfrak{g} \quad (260)$$

The commutator is $[\xi, \eta] = \xi^a \eta^b f_{ab}{}^c t_c$. Therefore the elements of \mathfrak{g}^* have lower indices, so the pairing of $X \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$ is $\langle X, \xi \rangle = X_a \xi^a$. The structure of a Lie algebra on \mathfrak{g} determines the differential on $\Lambda^\bullet \mathfrak{g}^*$, which is continued by the multilinearity from:

$$(dX)_{ab} = f_{ab}{}^c X_c \quad (261)$$

When B is decomposable, *i.e.* $B = b_1 \wedge b_2$, we can rewrite (259) as follows:

$$[X, Y]_B^{(1)} = \iota_{b_1} \text{ad}_{b_2}^* X \wedge Y \quad (262)$$

The Jacobi identity for $[\cdot, \cdot]_B^{(1)}$ gives:

$$[[X, Y]_B^{(1)}, Z]_B^{(1)} \pm (\text{cycl}) = (\iota_{b_1} \text{ad}_{b_2}^*)^2 (X \wedge Y \wedge Z) = \llbracket B, B \rrbracket^{abc} \iota_{t_a} \iota_{t_b} \text{ad}_{t_c}^* (X \wedge Y \wedge Z) \quad (263)$$

where the last equality is true also if B is not decomposable. This means that $[\cdot, \cdot]_B$ satisfies the Jacobi identity iff $\llbracket B, B \rrbracket = 0$.

But what happens if $\llbracket B, B \rrbracket$ is not zero, but is Q -exact in the sense of (210)? Then $\llbracket B, B \rrbracket^{abc} = A^{[a|m} f_m{}^{bc]}$ and we get:

$$\begin{aligned} \langle [[X, Y]_B^{(1)}, Z]_B^{(1)} \pm (\text{cycl. } X, Y, Z), t_q \rangle &= 6A^{[a|m} f_m{}^{bc]} X_{[a} Y_b Z_{c']} f_{cq}^{c'} = \\ &= 4A^{am} f_m{}^{bc} X_{[a} Y_b Z_{c']} f_{cq}^{c'} + 2A^{cm} f_m{}^{ab} X_{[a} Y_b Z_{c']} f_{cq}^{c'} = \\ &= 4A^{am} f_m{}^{[b|c]} X_{[a} Y_b Z_{c']} f_{cq}^{c'} + 2A^{cm} f_m{}^{ab} X_{[a} Y_b Z_{c']} f_{cq}^{c'} = \\ &= 2A^{am} f_{mq}{}^c X_{[a} Y_b Z_{c']} f_{c'}{}^b + 2A^{cm} f_m{}^{ab} X_{[a} Y_b Z_{c']} f_{cq}^{c'} = \\ &= \frac{1}{3} A^{am} f_{mq}{}^c X_a [Y, Z]_c + \frac{1}{3} A^{cm} [X, Y]_m Z_{c'} f_{cq}^{c'} \pm (\text{cycl. } X, Y, Z) \end{aligned} \quad (264)$$

where $[X, Y]_a = f_a{}^{bc} X_b Y_c$ (so defined $[X, Y]$ is a bracket on \mathfrak{g}^* which turns \mathfrak{g}^* into a Lie algebra isomorphic to \mathfrak{g}). Suppose that A is antisymmetric: $A^{am} = -A^{ma}$. In this case, let us define a new operation $[\cdot, \cdot]_A^{(2)} : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$; in coordinates:

$$\begin{aligned} ([X, Y]_A^{(2)})_q &= A^{am} X_a f_{mq}{}^b Y_b + A^{cm} f_{cq}{}^a X_a Y_m - (X \leftrightarrow Y) = \\ &= \langle (X \wedge Y), \llbracket t_q, A \rrbracket \rangle \end{aligned} \quad (265)$$

Conclusion If $\llbracket B, B \rrbracket$ is Q -exact in the sense of (210) with antisymmetric A then the following bracket on \mathfrak{g}^* :

$$[,] + \varepsilon[,]_B^{(1)} + \varepsilon^2[,]_A^{(2)} \quad (266)$$

satisfies the Jacobi identity up to the order ε^2 .

8.3 The space of solutions to $\llbracket B, B \rrbracket = 0$.

Unfortunately we do not have an explicit description of the space of solutions to $\llbracket B, B \rrbracket = 0$. Here we will discuss an subspace which corresponds to the Maldacena-Lunin solution. Then we will argue that this subspace does not exhaust all the solutions. In other words, there are beta-deformations other than the Maldacena-Lunin solution.

8.3.1 The solutions of Maldacena-Lunin type

Let us introduce the basis in $gl(m|n)$ consisting of the $m|n$ matrices E_j^i , which have 0 in all positions except for 1 in the i -th row and j -th column. For example, for $E_3^2 \in gl(3)$ is this:

$$E_3^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that E_j^j is a diagonal matrix.

It is straightforward to see that these matrices satisfy $\llbracket B, B \rrbracket = 0$:

$$B = \sum_{1 \leq i < j \leq 8} h^{ij} E_i^i \wedge E_j^j \quad (267)$$

In fact, for such B even stronger identity is true:

$$B^{ae} f_{eg}{}^b B^{gc} = 0 \quad (268)$$

— this is true even if we do not antisymmetrize a, b, c .

8.3.2 Solutions of more general type

Let us consider the deformations of (267) of the following type:

$$\begin{aligned} B &\rightarrow B + \delta B \\ \delta B &= \sum a_{ij} E_j^i \wedge E_i^j \end{aligned} \quad (269)$$

Notice that $\llbracket B, \delta B \rrbracket = 0$, but $B^{ae} f_{eg}{}^b \delta B^{gc}$ is nonzero. If this deformation remains unobstructed at higher orders, then we must conclude that $\llbracket B, B \rrbracket = 0$ *does not* imply (268). As we discussed in Section 1.3.1 the existing mathematical results on the classification of solutions of the classical Yang-Baxter equation seem to imply that indeed there are solutions of $\llbracket B, B \rrbracket = 0$ which do not imply (268).

8.4 Calculation of the bracket for elements of $6_{\mathbf{C}}^{u(3)} \subset 45_{\mathbf{C}}^{su(4)}$

The B -tensors corresponding to $6_{\mathbf{C}}^{u(3)} \subset 45_{\mathbf{C}}^{su(4)}$ are of the following form:

$$B_{jl}^{ik} = u^{ikp} \epsilon_{pjl} + u_{pjl}^* \epsilon^{pik} \quad (270)$$

We have explicitly calculated $[[B, B]]$ for such B :

$$\begin{aligned} [[B, B]]_{\mathbf{jln}}^{\mathbf{ikm}} = & -2 \delta_1^{\mathbf{i}} u^{\mathbf{kmp}} u_{\mathbf{jnp}}^* + 2 \epsilon^{\mathbf{ikp}} \epsilon_{\mathbf{lnq}} u^{r\mathbf{mq}} u_{\mathbf{jrp}}^* + \\ & + 2 u^{\mathbf{ikp}} \epsilon_{\mathbf{jrp}} u^{r\mathbf{mq}} \epsilon_{\mathbf{lnq}} + 2 \epsilon^{\mathbf{ikp}} u_{\mathbf{jrp}}^* \epsilon^{r\mathbf{mq}} u_{\mathbf{lnq}}^* + \\ & + \left[\begin{array}{l} \text{terms dictated} \\ \text{by antisymmetry} \end{array} \quad \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathbf{k} \\ \mathbf{l} \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix} \right] \end{aligned} \quad (271)$$

(we boldfaced the uncontracted indices for convenience).

9 Complex β -deformation

9.1 Complex structure in $45_{\mathbf{C}}$

There is an injective map from $45_{\mathbf{C}}^{su(4)}$ to the space of B -fields on S^5 . For each $B \in 45_{\mathbf{C}}^{su(4)}$ we get the corresponding B -field which we call \mathcal{B} and which is given by Eq. (310):

$$\mathcal{B} = B^{ab} \text{tr}((dgg^{-1})_2 g t_a g^{-1}) \wedge \text{tr}((dgg^{-1})_2 g t_b g^{-1}) \quad (272)$$

or, more explicitly, in terms of the coordinates X^a describing the embedding of S^5 into \mathbf{R}^6 by Eq. (77):

$$\mathcal{B} = B^{[kl][mn]} X_{[k} dX_{l]} \wedge X_{[m} dX_{n]} \quad (273)$$

This shows that the space of 2-forms on S^5 has a subspace which transforms as $45_{\mathbf{C}}$ under $so(6)$. We would like to stress that these B -fields are *real*; but the representation in which they transform happens to have a complex structure; therefore the subindex \mathbf{C} in $45_{\mathbf{C}}$. This complex structure is explained in Section A, but here we want to discuss a physical explanation of it.

It turns out that if the field \mathcal{B} is defined by (273) then $*_5 \mathcal{B}$ is exact:

$$*_5 \mathcal{B} = d\tilde{\mathcal{B}} \quad (274)$$

Indeed, notice that $*_5 \mathcal{B}$ is closed. Indeed, $*_5 d *_5 \mathcal{B}$ is a 1-form on S^5 , in other words:

$$*_5 d *_5 \mathcal{B} \in \text{ind}_{so(5)}^{so(6)} \mathbf{R}^5 \quad (275)$$

But there is no $45_{\mathbf{C}}^{so(6)}$ in $\text{ind}_{so(5)}^{so(6)} \mathbf{R}^5$:

$$\text{Hom}_{so(6)} \left(45_{\mathbf{C}}, \text{ind}_{so(5)}^{so(6)} \mathbf{R}^5 \right) = \text{Hom}_{so(5)} \left(\frac{(\mathbf{R}^5 \wedge \mathbf{R}^5 \oplus \mathbf{R}^5)^{\wedge 2}}{\mathbf{R}^5 \wedge \mathbf{R}^5 \oplus \mathbf{R}^5}, \mathbf{R}^5 \right) = 0 \quad (276)$$

and this shows that $d *_5 \mathcal{B} = 0$. Strictly speaking Eq. (274) only defines $\tilde{\mathcal{B}}$ up to a total derivative. But this ambiguity is fixed by the requirement that the correspondence $\mathcal{B} \mapsto \tilde{\mathcal{B}}$ commutes with the symmetries. We can demonstrate this in the following way. Let us start by fixing B so that:

$$\square_6 \mathcal{B} = 0 \quad (277)$$

Then we get:

$$*_5 d *_5 d \mathcal{B} = \iota_E *_6 d \iota_E *_6 d \mathcal{B} = \iota_E *_6 \mathcal{L}_E *_6 d \mathcal{B} = -16 \mathcal{B} \quad (278)$$

Let us therefore denote:

$$\mathcal{I} \mathcal{B} = \frac{1}{4} *_5 d \mathcal{B} \quad (279)$$

The operator \mathcal{I} is the complex structure, $\mathcal{I}^2 = -1$. We have:

$$*_5 \mathcal{B} = d \left(-\frac{1}{4} \mathcal{I} \mathcal{B} \right) \quad (280)$$

Roughly speaking, the complex structure exchanges the NSNS and the RR B -fields.

9.2 Comparison with the quadratic obstruction found by Aharony, Kol, and Yankielowicz [1]

9.2.1 Formula for obstruction suggested in [1]

The deformations considered in [1] correspond on the Yang-Mills side to the following deformation of the superpotential:

$$W = \frac{1}{3} h_{ijk} \text{tr}(\Phi^i \Phi^j \Phi^k) \quad (281)$$

The coefficients h_{ijk} transform in the symmetric tensor product of three fundamental representations of $u(3) \subset so(6)$. In [1] the following condition on h_{ijk} was obtained:

$$h_{ipq} \bar{h}^{jpq} - \frac{1}{3} \delta_i^j h_{pqr} \bar{h}^{pqr} = 0 \quad (282)$$

This is Eq. (3.5) in [1].

9.2.2 Comparison with $[[B, B]]$

To compare with our results we consider the case when B is in $6_{\mathbf{C}}^{u(3)} \subset 45_{\mathbf{C}}^{su(4)}$. We have considered this case in Section 8.4. Notice that (282) is much weaker than the condition of vanishing of $[[B, B]]$ which on $6_{\mathbf{C}}^{u(3)}$ is given by (271). Indeed, the complex β -deformations do not satisfy $[[B, B]] = 0$, and not even (11). As we have explained in Section 7.4, we have to take into account the possibility that $[[B, B]]$ is nonzero but $[[B, B]]^{abc} \Lambda_a \Lambda_b \Lambda_c$ is Q_0 -exact. The careful analysis of the SUGRA equations presented in [1] shows that the log terms (the resonant terms, corresponding to the anomalous dimension) actually appear

only at the third order in ϵ . This suggests that $Q_0^{-1} \llbracket B, B \rrbracket^{abc} \Lambda_a \Lambda_b \Lambda_c$ exists and does not contain log terms, although it might be not strictly covariant (maybe only covariant up to $\text{Ker } Q_0$).

9.2.3 Supersymmetric extension of (282)

It is natural to ask the following questions:

1. what is the supersymmetric extension of (282)?
2. is it possible to express it in terms of the Schouten bracket $\llbracket B, B \rrbracket$?

The Schouten bracket $\llbracket B, B \rrbracket$ is an element of $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. But let us *pick a representative* \widehat{B} for B in $\widehat{\mathfrak{g}} = su(2, 2|4)$ and consider $\llbracket \widehat{B}, \widehat{B} \rrbracket$ as an element of $\widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}}$. Notice that $\widehat{\mathfrak{g}}$ is a subspace in the tensor product of the fundamental and the antifundamental representation, defined by the tracelessness condition. Therefore we parametrize $x \in \widehat{\mathfrak{g}}$ by a $(4|4) \times (4|4)$ -matrix x_b^a ; the upper index parametrizes the fundamental representation, and the lower index the antifundamental. With these notations a tensor $X \in \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}}$ has six indices X_{bdf}^{ace} . Let us consider the following contraction:

$$| \llbracket \widehat{B}, \widehat{B} \rrbracket |_b^a \stackrel{\text{def}}{=} \llbracket \widehat{B}, \widehat{B} \rrbracket_{pqb}^{apq} \quad (283)$$

It turns out that $| \llbracket \widehat{B}, \widehat{B} \rrbracket |$ does not change if we change $\widehat{B} \mapsto \widehat{B} + \mathbf{1} \wedge x$ (*i.e.* pick a different representative for B in $\widehat{\mathfrak{g}}$), and also does not change if we change $B^{ab} \mapsto f^{ab}_c G^c$. There is a simplified formula for $| \llbracket \widehat{B}, \widehat{B} \rrbracket |$:

$$| \llbracket \widehat{B}, \widehat{B} \rrbracket |_b^a = \widehat{B}_{qr}^{pa} \widehat{B}_{pb}^{rq} \quad (284)$$

A direct calculation shows that $| \llbracket \widehat{B}, \widehat{B} \rrbracket |_b^a$ is invariant under $\widehat{B}_{cd}^{ab} \mapsto \widehat{B}_{cd}^{ab} + \delta_c^a X_d^b - \delta_d^a X_c^b$ and under $\widehat{B}_{cd}^{ab} \mapsto \widehat{B}_{cd}^{ab} + \delta_d^a G_c^b - \delta_c^a G_d^b$. Therefore $| \llbracket \widehat{B}, \widehat{B} \rrbracket |_b^a$ can be considered a function of B , and moreover is correctly defined on the equivalence classes of B with respect to (69). Also, it follows from symmetries that $| \llbracket \widehat{B}, \widehat{B} \rrbracket |$ is traceless:

$$| \llbracket \widehat{B}, \widehat{B} \rrbracket |_a^a = 0 \quad (285)$$

Also:

$$| \llbracket \widehat{B}, I\widehat{B} \rrbracket | = 0 \quad (286)$$

Moreover, one can check that when $B \in \mathbf{6}_C$:

$$\frac{1}{6} | \llbracket \widehat{B}, \widehat{B} \rrbracket |_i^j = u_{ipq} \bar{u}^{jpq} - \frac{1}{3} \delta_i^j h_{pqr} \bar{h}^{pqr} \quad (287)$$

This means that the following condition:

$$| \llbracket \widehat{B}, \widehat{B} \rrbracket |_b^a = 0 \quad (288)$$

is the supersymmetric version of Eq. (282).

Let us now return to the question: “is it possible to formulate the supersymmetric analogue of the condition (282) suggested in [1] in terms of the Schouten bracket $[[B, B]]$?” We find ourselves in the following interesting situation. Our condition (288) is expressed in terms of the Schouten bracket in $\widehat{\mathfrak{g}}$, which requires the choice of \widehat{B} , which is a lift of B from \mathfrak{g} up to $\widehat{\mathfrak{g}}$. The projection of the bracket $|[[\widehat{B}, \widehat{B}]]|$ does not depend on the choice of \widehat{B} for a given B , and therefore is a well defined quadratic function of B . But the bracket $[[\widehat{B}, \widehat{B}]]$ itself does depend on the choice of a lift. Notice that our $[[\widehat{B}, \widehat{B}]]$ takes values¹² in $\widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}}$ rather than $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. Then we apply the “projection” $|\dots|$. This projection is defined on $\widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}}$:

$$|\dots| : \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}' \quad (289)$$

But there is no such operation as $|\dots|$ on $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ because the operation $|x \wedge y \wedge z|$ is not correctly defined with respect to the identification $x_b^a \simeq x_b^a + c\delta_b^a$.

10 Reading the supergravity fields from the vertex

In this section we will restrict ourselves to the case when the nonzero components of B^{ab} are only in the direction of S^5 . We will consider the deformed action (74) and read the supergravity fields from this action.

The only nonzero perturbations at the first order in the deformation parameter are the NSNS and RR B -fields. Both can be read from the deformed action (74). We will start in Section 10.1 with briefly reviewing the the general action of Berkovits and Howe [25]. We then compare our deformed action (74) with their general action. In Section 10.2 we consider the “bosonic part of the action”, *i.e.* the action at $\theta = 0$, and read the NSNS B -field from the term $B_{mn}\partial x^m \bar{\partial} x^n$. The measurement of the RR B -field is a bit more subtle, it requires the analysis of the fermionic terms in the action. We discuss the fermionic terms and the RR B -field in Section 10.3.

¹² When we calculate $[[\ , \]]$ on B we consider it taking values in $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. But when we calculate $[[\ , \]]$ on \widehat{B} we consider it taking values in $\widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}} \wedge \widehat{\mathfrak{g}}$. In the first case, this is the Schouten bracket on $\Lambda^\bullet \mathfrak{g}$, and in the second case this is the Schouten bracket on $\Lambda^\bullet \widehat{\mathfrak{g}}$. Both are denoted $[[\ , \]]$.

10.1 The action of Berkovits and Howe

Brief partial review of [25] The most general action with BRST symmetry is [25]:

$$\begin{aligned}
S = \frac{1}{2\pi\alpha'} \int d^2z \bigg(& \frac{1}{2} (G_{MN}(Z) + B_{MN}(Z)) \partial Z^M \bar{\partial} Z^N + E_M^\alpha(Z) d_\alpha \bar{\partial} Z^M + \\
& + E_M^{\hat{\alpha}}(Z) \tilde{d}_{\hat{\alpha}} \partial Z^M + \Omega_{M\alpha}^\beta(Z) \lambda^\alpha w_\beta \bar{\partial} Z^M + \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}(Z) \tilde{\lambda}^{\hat{\alpha}} \tilde{w}_{\hat{\beta}} \partial Z^M + \\
& + P^{\alpha\hat{\beta}}(Z) d_\alpha \tilde{d}_{\hat{\beta}} + C_\alpha^{\beta\hat{\gamma}}(Z) \lambda^\alpha w_\beta \tilde{d}_{\hat{\gamma}} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(Z) \tilde{\lambda}^{\hat{\alpha}} \tilde{w}_{\hat{\beta}} d_\gamma + \\
& + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}(Z) \lambda^\alpha w_\beta \tilde{\lambda}^{\hat{\gamma}} \tilde{w}_{\hat{\delta}} + \frac{1}{2} \alpha' \Phi(Z) r \bigg) + S_\lambda + S_{\tilde{\lambda}} \quad (290)
\end{aligned}$$

In this action Z^M are the coordinates of the $(10|32)$ -dimensional supermanifold (the space-time). They are the *worldsheet* fields, *i.e.* $Z^M = Z^M(z, \bar{z})$. There are also the pure spinor worldsheet fields $\lambda(z, \bar{z}), \tilde{\lambda}(z, \bar{z}), w(z, \bar{z}), \tilde{w}(z, \bar{z})$ and the worldsheet auxiliary fields $d(z, \bar{z}), \tilde{d}(z, \bar{z})$. It is important that fields λ, w, d are of two kinds: “left” and “right”, for example λ and $\tilde{\lambda}$. They become left- and right-movers in the flat space limit. The “right” fields are marked by tilde. There is a left ghost number $U(1)_L$ under which λ^α has charge 1 and $\tilde{\lambda}^{\hat{\alpha}}$ charge 0, and a similar right ghost number $U(1)_R$.

In the general curved spacetime there is no separation of the worldsheet dynamics into “left” and “right” sectors, like it was in flat space. But some aspects of it survive in curved spacetime. There is a separate left and right ghost number. The conserved current corresponding to the left BRST transformation is holomorphic, and the right one is antiholomorphic.

The pure spinor variables λ and their momenta w take values in the spin bundle, therefore greek letters are used for their indices. The *target space* fields are:

$$G_{MN}, B_{MN}, E_M^\alpha, E_M^{\hat{\alpha}}, \Omega_{M\alpha}^\beta, \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}, P^{\alpha\hat{\beta}}, C_\alpha^{\beta\hat{\gamma}}, \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}, S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}, \Phi \quad (291)$$

The leading components of the superfields $\Omega_{M\alpha}^\beta$ and $\hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}$ are the spin connections. The space-time metric G_{MN} is degenerate, of the rank $(10|0)$. (A non-degenerate metric would have rank $(10|32)$.) The BRST transformations act on the coordinate fields Z^M in the following way:

$$QZ^M = \lambda^\alpha E_\alpha^M + \tilde{\lambda}^{\hat{\alpha}} E_{\hat{\alpha}}^M \quad (292)$$

where E_α^M and $E_{\hat{\alpha}}^M$ are vector fields which span the kernel of the metric:

$$E_\alpha^M G_{MN} = E_{\hat{\alpha}}^M G_{MN} = 0 \quad (293)$$

and also satisfy the following equations:

$$E_M^\alpha E_\beta^M = \delta_\beta^\alpha, \quad E_M^\alpha E_{\hat{\beta}}^M = 0, \quad E_M^{\hat{\alpha}} E_\beta^M = 0, \quad E_M^{\hat{\alpha}} E_{\hat{\beta}}^M = \delta_{\hat{\beta}}^{\hat{\alpha}} \quad (294)$$

(Therefore, E_α^M and $E_{\hat{\alpha}}^M$ are determined in terms of the basic fields (291).)

The metric G_{MN} is degenerate, but the degeneracy is non-physical. It is non-physical because the vector field corresponding to Q_{BRST} traverses the kernel of G_{MN} as λ and $\tilde{\lambda}$ traverse the pure spinor cones. In this sense, the degeneration is BRST trivial.

After integrating out the auxiliary fields d and \tilde{d} the effective metric becomes non-degenerate:

$$G_{MN} - 2E_{(M}^\alpha P_{\alpha\hat{\beta}} E_{N)}^{\hat{\beta}} \quad (295)$$

The pure spinor kinetic terms S_λ and $S_{\tilde{\lambda}}$ are the same as in flat space:

$$S_\lambda = \int d^2z (w_{\alpha+}, \partial_- \lambda^\alpha) \ , \quad S_{\tilde{\lambda}} = \int d^2z (\tilde{w}_{\hat{\alpha}-}, \partial_+ \tilde{\lambda}^{\hat{\alpha}}) \quad (296)$$

Since λ and $\tilde{\lambda}$ are constrained to satisfy $\lambda^\alpha \Gamma_{\alpha\beta}^m \lambda^\beta = \tilde{\lambda}^{\hat{\alpha}} \Gamma_{\hat{\alpha}\hat{\beta}}^m \tilde{\lambda}^{\hat{\beta}} = 0$, the conjugate momenta $w_{\alpha+}$ and $\tilde{w}_{\hat{\alpha}-}$ are subject to the following gauge transformations with arbitrary vector parameters k_{m+} and \hat{k}_{m-} :

$$\delta_k w_{\alpha+} = k_{m+} \Gamma_{\alpha\beta}^m \lambda^\beta \ , \quad \delta_{\hat{k}} \tilde{w}_{\hat{\alpha}-} = \hat{k}_{m-} \Gamma_{\hat{\alpha}\hat{\beta}}^m \tilde{\lambda}^{\hat{\beta}} \quad (297)$$

The kinetic terms (296) are preserved by the separate $so(1,9)$ rotations of “left” and “right” (λ, w) :

$$\begin{aligned} \delta \lambda^\alpha &= A_\beta^\alpha \lambda^\beta \ , \quad \delta w_{\alpha+} = -A_\alpha^\beta w_{\beta+} \\ \delta \tilde{\lambda}_1^{\hat{\alpha}} &= \hat{A}_{\hat{\beta}}^{\hat{\alpha}} \tilde{\lambda}_1^{\hat{\beta}} \ , \quad \delta \tilde{w}_{\hat{\alpha}-} = -\hat{A}_{\hat{\alpha}}^{\hat{\beta}} \tilde{w}_{\hat{\beta}-} \end{aligned} \quad (298)$$

which act as gauge transformations on E_M^α , $E_M^{\hat{\alpha}}$, $\Omega_{M\alpha}^\beta$, $\hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}$, $P^{\alpha\hat{\beta}}$, $C_\alpha^{\beta\hat{\gamma}}$, $\hat{C}_{\hat{\alpha}}^{\hat{\beta}\hat{\gamma}}$, $S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}$. In this sense, there are two independent local $so(1,9)$ symmetries A and \hat{A} , *i.e.* $so(1,9) \oplus so(1,9)$.

At first sight it might seem that we can rotate with A and \hat{A} in $gl(16, \mathbf{C})$; the kinetic terms (296) would be invariant. However only the transformations with A and \hat{A} in $so(1,9) \subset gl(16, \mathbf{C})$ respect the gauge transformations (297).

The two Lorentz symmetries (298) are field redefinitions on the worldsheet, which look like gauge transformations from the target space point of view. In the target space $E_\alpha^M, E_{\hat{\alpha}}^M$ correspond to the basis in the tangent space to $\text{Ker } G$. Therefore the field redefinition (298) corresponds geometrically to the change of a basis in $\text{Ker } G$ consistent with the action of the $U(1)_{L-R}$ ghost symmetry (*i.e.* E_α^M cannot be mixed with $E_{\hat{\alpha}}^M$).

The authors of [25] suggested to partially fix this gauge symmetry, down to the conventional Lorentz gauge symmetry of the gravity theory:

$$so(1,9) \oplus so(1,9) \longrightarrow so(1,9) \quad (299)$$

They did it in the following way. Note that $\text{Ker } G$ is not an integrable distribution, there is a Frobenius map:

$$[,] : \text{Ker } G \wedge \text{Ker } G \rightarrow \text{Im } G \quad (300)$$

This map is given by the commutator of the vector fields; it is also called “torsion”. Note that $\text{Im } G$ has an orthogonal structure (given by G itself). Let us therefore fix some orthogonal basis in $\text{Im } G$; the ambiguity of such a fix is parametrized by $so(1, 9)$ in (299). Let us denote such a basis E_m^M . Here the vector index m runs from 0 to 9. Then, the partial gauge fixing of [25] consists of requiring the Frobenius map to satisfy¹³:

$$[\epsilon E_\alpha, \epsilon' E_\beta]^M = \epsilon \epsilon' \Gamma_{\alpha\beta}^m E_m^M \quad (301)$$

It is a nontrivial fact, proven in [25], that this choice of gauge is possible. We will discuss this gauge fixing specifically for our example of β -deformation in Section 10.3.4.

RR bispinor The leading component of $P^{\alpha\hat{\beta}}$ is the bispinor of the RR field strengths:

$$P^{\alpha\hat{\beta}} = F_m (\Gamma^m)^{\alpha\hat{\beta}} + F_{klm} (\Gamma^{klm})^{\alpha\hat{\beta}} + F_{klmnp} (\Gamma^{klmnp})^{\alpha\hat{\beta}} \quad (302)$$

Notice that these gauge transformations act on $P^{\alpha\hat{\beta}}$ and that it has a nonzero constant “background” value in the pure $AdS_5 \times S^5$:

$$P_0^{\alpha\hat{\beta}} = F_{klmnp}^{(0)} (\Gamma^{klmnp})^{\alpha\hat{\beta}} \quad (303)$$

where $F^{(0)}$ is the RR flux of the pure $AdS_5 \times S^5$. Since the background value $P_0^{\alpha\hat{\beta}}$ transforms nontrivially under the two local Lorentz transformations, we have to be careful in defining the fluctuation. When we write (302) we have to explain how we break the gauge from $so(1, 9) \oplus so(1, 9)$ down to the diagonal $so(1, 9)$. This is discussed in Section 10.3.4.

Notations specific to $AdS_5 \times S^5$ Because we expand around AdS, rather than flat space, we will choose a normalization of w and λ which is slightly different from what Berkovits and Howe used. We use the kinetic term:

$$S_{\lambda_3} = \int d^2 z \text{Str} (w_{1+} \partial_- \lambda_3) \quad , \quad S_{\lambda_1} = \int d^2 z \text{Str} (w_{3-} \partial_+ \lambda_1) \quad (304)$$

rather than (296). In the flat space notations our kinetic terms (304) gives instead of (296) the following expression:

$$S_\lambda = \int d^2 z (w_+^{\hat{\alpha}}, (\Gamma_{56789} \partial_- \lambda)_{\hat{\alpha}}) \quad (305)$$

$$S_{\tilde{\lambda}} = \int d^2 z (\tilde{w}_-^\alpha, (\Gamma_{56789} \partial_+ \tilde{\lambda})_\alpha) \quad (306)$$

¹³We multiplied the odd vector fields E_α and E_β by formal anticommuting parameters ϵ, ϵ' , to ease the notations

where Γ_{56789} is the product of gamma-matrices in the directions tangent¹⁴ to S^5 . The relation between (304) and (305), (306) can be summarized in the formula:

$$\text{Str}(w_1 \lambda_3) = (w_1^{\hat{\alpha}} C_{\hat{\alpha}\beta} \lambda_3^{\beta}) \quad (307)$$

where $C_{\hat{\alpha}\beta}$ is some bispinor. In fact $C_{\alpha\hat{\beta}}$ the inverse of the Ramond-Ramond bispinor field $P_0^{\alpha\hat{\beta}}$ in $AdS_5 \times S^5$. In agreement with (290) we denote $P^{\alpha\hat{\beta}}$ the Ramond-Ramond bispinor field, and $P_0^{\alpha\hat{\beta}}$ the background value of $P^{\alpha\hat{\beta}}$ in the pure $AdS_5 \times S^5$.

10.2 NS-NS B -field

Measuring the NSNS B -field The NS-NS B field can be read from (74) in a rather straightforward way, from the bosonic ($\theta = 0$) terms in the deformed action. Notice that the bosonic part of the undeformed action (44) is the $\theta = 0$ part of $\frac{1}{2} \int \text{Str}(J_{2+} J_{2-})$; it is parity-even (*i.e.* invariant under $+\leftrightarrow -$). This means, of course, that pure $AdS_5 \times S^5$ has zero NSNS B field. Let us now consider the deformation (74):

$$S \longrightarrow S + \frac{1}{2} B^{ab} \int j_{[a} \wedge j_{b]}$$

The $\theta = 0$ part of j_a is $2 \text{Str}((dgg^{-1})_2 g t_a g^{-1})$. Therefore the $j \wedge j$ term gives a parity-odd piece:

$$2 \int B^{ab} \text{tr}((dgg^{-1})_2 g t_a g^{-1}) \wedge \text{tr}((dgg^{-1})_2 g t_b g^{-1}) + \dots \quad (308)$$

which corresponds to the NSNS B -field. Unfortunately we have a conflict of notations: the letter B already stands for the deformation parameter B^{ab} . To avoid confusion we will denote the NSNS B field with the calligraphic letter \mathcal{B} . Eq. (308) implies:

$$\mathcal{B}_{\mu\nu} = B^{ab} \text{Str}(t_{\mu}^2 g t_a g^{-1}) \text{Str}(t_{\nu}^2 g t_b g^{-1}) \quad (309)$$

In particular when B^{ab} is tangent to the sphere, there is another way to write this:

$$\mathcal{B}_{[\mu\nu]} = B^{ab} [(g t_a g^{-1})_2, (g t_b g^{-1})_2]_{[\mu\nu]} \quad (310)$$

Comparing with the formulas of Maldacena and Lunin We should compare with the formula from [12]:

$$B = \mu_1^2 \mu_2^2 d\phi_1 \wedge d\phi_2 + \mu_2^2 \mu_3^2 d\phi_2 \wedge d\phi_3 + \mu_3^2 \mu_1^2 d\phi_3 \wedge d\phi_1 \quad (311)$$

¹⁴We use $\{0, \dots, 4\}$ to enumerate the tangent space directions of AdS_5 , and $\{5, \dots, 9\}$ for the tangent space directions of S^5 ; this should not be confused with the enumeration of X_1, \dots, X_6 in (312), where X_i are the flat coordinates of the auxiliary \mathbf{R}^6 which we use to embed $S^5 \subset \mathbf{R}^6$.

where μ and ϕ are described in Eq. ((3.11) of [12]) which in our language translates:

$$\begin{aligned} X_1 &= \mu_1 \cos \phi_1, & X_2 &= \mu_1 \sin \phi_1 \\ X_3 &= \mu_2 \cos \phi_2, & X_4 &= \mu_2 \sin \phi_2 \end{aligned} \quad (312)$$

$$X_5 = \mu_3 \cos \phi_3, \quad X_6 = \mu_3 \sin \phi_3 \quad (313)$$

Here X_1, \dots, X_6 give the embedding of S^5 into \mathbf{R}^6 : $X_1^2 + X_2^2 + \dots + X_6^2 = 1$ In terms of this embedding the bosonic part of the global conserved current is:

$$j_{[AB]} = 2(dg g^{-1})_{[AB]} = 2X_{[A} dX_{B]} \quad (314)$$

It follows that indeed (311) is of the form:

$$B^{[kl][mn]} X_{[k} dX_{l]} \wedge X_{[m} dX_{n]} \quad (315)$$

where $B^{[kl][mn]}$ has the following nonvanishing components:

$$B^{[12][34]} = B^{[34][56]} = B^{[56][12]} = 1 \quad (316)$$

(And those which follow from them by the antisymmetry of B , *e.g.* $B^{[34][12]} = -1$.) This is in agreement with our Eq. (310).

10.3 RR B -field

The RR field is more subtle to reconstruct. It requires the analysis of the fermionic terms in the action. An additional complication is the nonzero value of the background RR 5-form in $AdS_5 \times S^5$. This implies that the derivation of the RR 3-form from the fermionic terms in the action requires a careful treatment of the local symmetry breaking.

10.3.1 How we measure the RR field strength

The general formula for the BRST charges is:

$$Q_L = \int d\tau^+ \epsilon \lambda^\alpha d_{\alpha+}, \quad Q_R = - \int d\tau^- \epsilon \tilde{\lambda}^{\hat{\alpha}} \tilde{d}_{\hat{\alpha}-} \quad (317)$$

where $d_{\alpha+}$ and $\hat{d}_{\hat{\alpha}-}$ are some composite fields, which are interpreted as auxiliary fields in [25]. We have denoted them j_{L+} and j_{R-} in Section 7.2:

$$j_{L+} = d_+, \quad j_{R-} = d_- \quad (318)$$

Integrating out d_α in (290) gives:

$$d_{\alpha+} = P_{\alpha\hat{\beta}} E_M^{\hat{\beta}} \partial_+ Z^M + \dots \quad (319)$$

where ... stands for terms containing w and λ . On the other hand it follows from ((62) of [25]) that the BRST operator Q_R acts on the matter fields Z^M in the following way:

$$Q_R Z^M = \tilde{\lambda}^{\hat{\alpha}} E_{\hat{\alpha}}^M \quad (320)$$

It follows from (319) and (320) that:

$$d_{\alpha+}|_{\partial_+ Z^M \rightarrow Q_R Z^M} \stackrel{\text{substitute:}}{=} P_{\alpha\hat{\beta}} \tilde{\lambda}^{\hat{\beta}} \quad (321)$$

In other words:

$$\epsilon \lambda^\alpha P_{\alpha\hat{\beta}} \epsilon' \tilde{\lambda}^{\hat{\beta}} = \epsilon \lambda^\alpha d_{\alpha+}|_{\partial_+ Z^M \rightarrow \epsilon' Q_R Z^M} \stackrel{\text{substitute:}}{=} \quad (322)$$

Procedure for measuring the RR field strength:

1. Calculate $d_{\alpha+}$ using (318) and (188).
2. Calculate $P_{\alpha\hat{\beta}}$ from (322). In fact it will turn out that the $\theta = 0$ component of $P_{\alpha\hat{\beta}}$ is the same as in pure $AdS_5 \times S^5$.
3. Examine the torsion and fix the proper gauge, Section 10.3.4. At this stage we do some field redefinition, an $so(1,9)$ rotation of w and λ , and the rotation in the opposite direction of \tilde{w} and $\tilde{\lambda}$. This affects $P_{\alpha\hat{\beta}}$, generating a 3-form piece corresponding to the RR 3-form.

10.3.2 Steps 1 and 2: calculate the deformation of $P_{\alpha\hat{\beta}}$

From (322) and (188) we read:

$$\epsilon \lambda^\alpha P_{\alpha\hat{\beta}} \epsilon' \tilde{\lambda}^{\hat{\beta}} = \text{Str}(\epsilon' \lambda_1 \epsilon \lambda_3) - 16(g^{-1} \epsilon' \lambda_1 g)_a B^{ab} (g^{-1} \epsilon \lambda_3 g)_b \quad (323)$$

The $\theta = 0$ component of $P_{\alpha\hat{\beta}}$ is undeformed The first term $\text{Str}(\epsilon' \lambda_1 \epsilon \lambda_3)$ on the right hand side of (323) corresponds to the background RR 5-form in the undeformed $AdS_5 \times S^5$, and the second term $-16(g^{-1} \epsilon' \lambda_1 g)_a B^{ab} (g^{-1} \epsilon \lambda_3 g)_b$ is the deformation. But we observe that the $\theta = 0$ component of the deformation $-16(g^{-1} \epsilon' \lambda_1 g)_a B^{ab} (g^{-1} \epsilon \lambda_3 g)_b$ is zero¹⁵. Naively this would imply that the RR field strength is undeformed. However, for the purpose of measuring the RR field this $P_{\alpha\hat{\beta}}$ is *a priori* in the **wrong gauge**. To understand what is the proper gauge choice we will look at the torsion, in Section 10.3.4.

¹⁵Remember that in this section we are only consider the case when the only nonzero components of B^{ab} are those with a and b both bosonic indices, *i.e.* both a and b are in $\mathbf{g}_{even} = \mathbf{g}_0 + \mathbf{g}_2$. Therefore the expansion of $(g^{-1} \epsilon \lambda g)$ starts with $[\epsilon \lambda, \theta]$.

10.3.3 Digression: integrating in d and \hat{d}

(this section is not needed for the main line of argument)

In this paper we are using the formulation of the worldsheet theory without the auxiliary fields d, \hat{d} ; the difference with the Berkovits–Howe action (290) is that d and \hat{d} have been integrated out. In this section we explain how to restore, or “integrate in” these auxiliary fields, and present the action in the form (290).

Let us calculate $d_{\alpha+} P^{\alpha\hat{\beta}} d_{\hat{\beta}-}$. We have:

$$d_{1+} = J_{1+} + 4B^{ab} j_{a+} (gt_b g^{-1})_1 \quad (324)$$

$$d_{3-} = J_{3-} + 4B^{ab} j_{a-} (gt_b g^{-1})_3 \quad (325)$$

Combining this with (323) we get:

$$\begin{aligned} d_{\alpha+} P^{\alpha\hat{\beta}} d_{\hat{\beta}-} &= \text{Str}(J_{1+} J_{3-}) + 4B^{ab} j_{a+} (g^{-1} J_{3-} g)_b - 4B^{ab} (g^{-1} J_{1+} g)_a j_{b-} + \\ &\quad + 16B^{ab} (g^{-1} J_{1+} g)_a (g^{-1} J_{3-} g)_b \end{aligned} \quad (326)$$

Consider the Lagrangian with d and \hat{d} integrated in:

$$\begin{aligned} &\text{Str}\left(\frac{1}{2}J_{2+}J_{2-} + \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-}\right) + B^{ab}j_{a+}j_{b-} - \\ &- \text{Str}\left(J_{1+}J_{3-}\right) - 4B^{ab}j_{a+}(g^{-1}J_{3-}g)_b + 4B^{ab}(g^{-1}J_{1+}g)_a j_{b-} - 16B^{ab}(g^{-1}J_{1+}g)_a (g^{-1}J_{3-}g)_b + \\ &\quad + (\text{terms linear and quadratic in } d, \hat{d}) + (\text{terms with ghosts}) \end{aligned} \quad (327)$$

Let us denote $k_{1a} = (g^{-1}J_1g)_a$, $k_{2a} = (g^{-1}J_2g)_a$, and $k_{3a} = (g^{-1}J_3g)_a$. We get:

$$\begin{aligned} &\text{Str}\left(\frac{1}{2}J_{2+}J_{2-} - \frac{1}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-}\right) + \\ &\quad + B^{ab}k_{1a+}k_{1b-} + 2B^{ab}k_{1a+}k_{2b-} - B^{ab}k_{1a+}k_{3b-} - \\ &\quad - 2B^{ab}k_{2a+}k_{1b-} - 4B^{ab}k_{2a+}k_{2b-} + 2B^{ab}k_{2a+}k_{3b-} - \\ &\quad - B^{ab}k_{3a+}k_{1b-} - 2B^{ab}k_{3a+}k_{2b-} + B^{ab}k_{3a+}k_{3b-} \\ &\quad + (\text{terms linear and quadratic in } d, \hat{d}) + (\text{terms with ghosts}) \end{aligned} \quad (328)$$

Consider the parity-even part of the deformed action, *i.e.* the part symmetric under $(+ \leftrightarrow -)$. It is equal to:

$$\text{Str}\left(\frac{1}{2}J_{2+}J_{2-}\right) + 2B^{ab}(k_{1a+}k_{2b-} + k_{1a-}k_{2b+}) + 2B^{ab}(k_{2a+}k_{3b-} + k_{2a-}k_{3b+}) \quad (329)$$

This should be identified with the term $1/2G_{MN}\partial_+Z^M\partial_-Z^N$ in the action (290) of Berkovits and Howe. The corresponding deformed metric G_{MN} has an important property. Namely, it

remains degenerate to the first order in B^{ab} . It is important in [25] that G_{MN} is a degenerate metric. More precisely, it should have rank $(10|0)$ while the maximal possible rank would be $(10|32)$. If there was a term proportional to $B^{ab}(k_{1a+}k_{3b-} + k_{1a-}k_{3b+})$ in (329), then this would mean that the deformed metric is not sufficiently degenerate. But there is no such term, therefore the deformed metric is as degenerate as it should be according to [25].

10.3.4 Step 3: examine the torsion components $T_{\alpha\beta}^m$ and $\hat{T}_{\hat{\alpha}\hat{\beta}}^m$ and do the necessary field redefinitions of w and λ

The torsion is defined using the commutator of the covariant derivatives:

$$\{D_\alpha, D_\beta\} = T_{\alpha\beta}^m D_m + T_{\alpha\beta}^\gamma D_\gamma + T_{\alpha\beta}^{\hat{\gamma}} D_{\hat{\gamma}} + (\text{terms without derivative}) \quad (330)$$

In this formula the covariant derivatives D_α and $D_{\hat{\alpha}}$ can be read from the BRST transformation of the matter fields:

$$Q_{BRST} Z^M = (\lambda^\alpha D_\alpha + \lambda^{\hat{\alpha}} D_{\hat{\alpha}}) Z^M + \dots \quad (331)$$

where \dots stands for terms containing w and λ . As explained in [25], the worldsheet theory of (290) has three independent Lorentz gauge groups: one acting on the spinor indices $\alpha, \beta, \gamma, \dots$, the other acting on the hatted spinor indices $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \dots$, and the third one acting on the vector indices m, n, \dots . These three Lorentz gauge groups are fixed down to one “usual” Lorentz gauge group by requesting that the $T_{\alpha\beta}^m$ and $T_{\hat{\alpha}\hat{\beta}}^m$ are equal to the gamma-matrices. In our notations, they can be identified with the structure constants:

$$T_{\alpha\beta}^m = f_{\alpha\beta}^m \quad , \quad \hat{T}_{\hat{\alpha}\hat{\beta}}^m = f_{\hat{\alpha}\hat{\beta}}^m \quad (332)$$

It is *in this gauge* that the field $P^{\alpha\hat{\beta}}$ is identified with the Ramond-Ramond bispinor. Our action (3) is *not automatically* in this form. In fact, we have to make the infinitesimal field redefinitions:

$$\begin{aligned} \delta\lambda_3 &= [\Phi, \lambda_3] \quad , \quad \delta w_{1+} = [\Phi, w_{1+}] \\ \delta\tilde{\lambda}_1 &= -[\Phi, \tilde{\lambda}_1] \quad , \quad \delta\tilde{w}_{3-} = -[\Phi, \tilde{w}_{3-}] \end{aligned} \quad (333)$$

with some matrix Φ in order to satisfy (332). Indeed, let us calculate $T_{\alpha\beta}^m$ in our case using (330) and (331). Consider the action of $Q = Q_0 + \varepsilon Q_1$ on g . Let R_{t_a} denotes the the right shift $R_{t_a} g = g t_a$, then Eq. (178) implies:

$$Q = R_{g^{-1}\lambda_3 g} + 4\varepsilon \Lambda_a B^{ab} R_{t_b} \quad (334)$$

(We are here only interested in the action on g , so we only keep the first term in (178).) Therefore:

$$D_\alpha = R_{g^{-1}t_\alpha^3 g} + 4\varepsilon (g^{-1}t_\alpha^3 g)_a B^{ab} R_{t_b} \quad (335)$$

Notice that, when acting on scalar functions, $R_{t_b} \simeq R_{g^{-1}(gt_b g^{-1})_{\perp} g}$. If we look at the $\theta = 0$ component, $(gt_b g^{-1})_{\perp} = (gt_b g^{-1})_2$. (Remember that in this Section, B^{ab} has only bosonic indices.) This implies that:

$$\{D_{\alpha}, D_{\beta}\} = f_{\alpha\beta}{}^m (R_{g^{-1}t_m^2 g} + 8\varepsilon (g^{-1}t_m^2 g)_a B^{ab} R_{g^{-1}(gt_b g^{-1})_2 g}) \quad (336)$$

$$= f_{\alpha\beta}{}^m (R_{g^{-1}t_m^2 g} + 8\varepsilon (gt_a g^{-1})_m B^{ab} R_{g^{-1}(gt_b g^{-1})_2 g}) \quad (337)$$

We gauge away the term proportional to ε by rotating λ and w as in (333) with Φ given by:

$$\Phi = 4B^{ab}[(gt_a g^{-1})_2, (gt_b g^{-1})_2] \quad (338)$$

10.3.5 Conclusion: RR field strength is *-dual in TS^5 to NSNS B -field

We have two observations:

- Explicit evaluation of $P_{\alpha\hat{\beta}}$ in Section 10.3.2 shows that the $\theta = 0$ component is undeformed, **but**:
- The phase rotation (333) of w and λ with the parameter Φ given by (338) is needed to bring the torsion components to the canonical form. In a similar way, one can see that \tilde{w} and $\tilde{\lambda}$ should be rotated in the opposite direction, *i.e.* $\tilde{\Phi} = -\Phi$.

Comparing Eq. (338) to Eq. (310) of Section 10.2, we see that Φ is equal to the NSNS B field contracted with two gamma-matrices:

$$\Phi = 8 \iota(\Gamma)\iota(\Gamma)\mathcal{B} \quad (339)$$

As we explained in Section 10.1, this phase rotation corresponds to the local Lorentz transformation of the target space fields. After this Φ -rotation the $\theta = 0$ component of the bispinor field becomes:

$$P_{(0)} + 2\Phi P_{(0)} \quad (340)$$

Given the relation (302) between P and the RR field strengths this implies:

$$F_{m_1 m_2 m_3} = 16 \mathcal{B}^{pq} F_{pq m_1 m_2 m_3} \quad (341)$$

Therefore we conclude that the RR field strength is *-dual in the tangent space to S^5 to the NSNS B -field.

10.3.6 Digression: the coupling of $w\lambda$ to the NSNS 3-form

(this section is not needed for the main line of argument)

What do we expect the kinetic term to be? Consider the terms in (74) which are quadratic in the pure spinor variables w and λ . What can we say about the part of the action quadratic in pure spinors? Looking at (290) we should expect it to be of the form:

$$\int d^2z \left[(w_+(\partial_- + \Omega_-)\lambda) + (\tilde{w}_-(\partial_+ + \tilde{\Omega}_+)\tilde{\lambda}) \right] \quad (342)$$

where Ω is the spin connection, which is the sum of the “geometrical” spin connection and the NSNS field strength¹⁶ $H = d\mathcal{B}$ contracted with two gamma-matrices.

Notice that there is H_{NSNS} in the covariant derivative The reason for including the NSNS field strength inside the spin connection is because given the supergravity constraints derived by Berkovits and Howe as conditions for classical BRST invariance and the solution of those constraints, we can use the Bianchi identity $dH = 0$; more specifically the components $(\nabla H)_{ab\alpha\beta}$ of this Bianchi identity, to find the relations $T_{ab}{}^c = H_{abd}\eta^{dc}$ and $\tilde{T}_{ab}{}^c = -H_{abd}\eta^{dc}$, where $T_{ab}{}^c$ and $\tilde{T}_{ab}{}^c$ are the torsion components constructed with the spin connections Ω and $\tilde{\Omega}$ respectively. Now, given those relations between torsions and the NSNS field strengths, one can make the redefinitions

$$\begin{aligned} \Omega_{ab}{}^c &\rightarrow \Omega_{ab}{}^c - \frac{1}{2}H_{abd}\eta^{dc}, \\ \tilde{\Omega}_{ab}{}^c &\rightarrow \tilde{\Omega}_{ab}{}^c + \frac{1}{2}H_{abd}\eta^{dc}, \end{aligned} \quad (343)$$

which means that the bosonic components of the redefined torsions are set to zero. Also, we remind that because of the pure spinor condition, the spin connections which appear in the Berkovits and Howe action can be decomposed as

$$\begin{aligned} \Omega_{M\alpha}{}^\beta &= \Omega_M^{(s)}\delta_\alpha{}^\beta + \frac{1}{4}\Omega_{Mab}(\Gamma^{ab})_\alpha{}^\beta \\ \tilde{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}} &= \tilde{\Omega}_M^{(s)}\delta_{\hat{\alpha}}{}^{\hat{\beta}} + \frac{1}{4}\tilde{\Omega}_{Mab}(\Gamma^{ab})_{\hat{\alpha}}{}^{\hat{\beta}}. \end{aligned} \quad (344)$$

Furthermore, as argued in [26], the constraints $T_{a\alpha}{}^\alpha = T_{a\hat{\alpha}}{}^{\hat{\alpha}} = 0$ imply that $\Omega_a^{(s)} = \tilde{\Omega}_a^{(s)} = 0$. Then, this last fact, the replacements (343) and equations (344) replaced in (342) show that indeed there is a three-form in the connection contracted with two gamma matrices.

What we get from our construction There are terms coming from the undeformed action, pure $AdS_5 \times S^5$:

$$\int d^2z \text{Str} (w_{1+}D_{0-}\lambda_3 + w_{3-}D_{0+}\lambda_1) \quad (345)$$

and terms coming from the deformation term $\int V_1^{(2)}$:

$$\int d^2z B^{ab} \left(-4j_{a-}\text{Str}(\{w_{1+}, \lambda_3\}(gt_bg^{-1})_0) - 4j_{a+}\text{Str}(\{w_{3-}, \lambda_1\}(gt_bg^{-1})_0) \right) \quad (346)$$

¹⁶Recall that we are using a calligraphic letter \mathcal{B} for the NSNS B-field to distinguish it from the deformation parameter B ; see Eqs. (309) and (349).

Therefore, the part of the action quadratic in w and λ is the total (345) + (346). Let us consider (346), with the fermionic fields θ turned off. In this case j becomes $2g^{-1}(*dgg^{-1})_2g$ and (346) reads:

$$\int d^2z \quad 8B^{ab} \left(\text{Str}((\partial_- gg^{-1})_2 gt_ag^{-1}) \text{Str}(\{w_{1+}, \lambda_3\}gt_bg^{-1}) - \right. \\ \left. - \text{Str}((\partial_+ gg^{-1})_2 gt_ag^{-1}) \text{Str}(\{w_{3-}, \lambda_1\}gt_bg^{-1}) \right) \quad (347)$$

This has to be compared to (342). What do we expect Ω to be in (342)? We know that *the metric* is undeformed at the first order in the deformation parameter, therefore there should be no correction to the “geometrical” part of the spin connection. But *the B field* is nonzero and the corresponding 2-form $\mathcal{B}_{mn} dx^m \wedge dx^n$ is given by:

$$\mathcal{B} = 2B^{ab} \text{Str}((dgg^{-1})_2(gt_ag^{-1})) \wedge \text{Str}((dgg^{-1})_2(gt_bg^{-1})) \quad (348)$$

The field strength is:

$$H = d\mathcal{B} = -8 \text{Str}((dgg^{-1})_2(dgg^{-1})_2(gt_ag^{-1})_0) B^{ab} \text{Str}((dgg^{-1})_2(gt_bg^{-1})_2) \quad (349)$$

Obviously, H has three tangent space indices, corresponding to the three dgg^{-1} . The contraction of the tangent space indices with the gamma-matrices works as follows:

$$\Gamma^m \iota_{R_{t_m^2}} \quad \text{where } R_{t_m^2} \text{ is the right invariant vector field: } R_{t_m^2} g = t_m^2 g \quad (350)$$

In other words, the contraction with the gamma-matrices can be schematically presented as the following rule:

$$dgg^{-1} \mapsto \Gamma^m \otimes t_m^2 \quad (351)$$

For example, let us contract $\text{tr}((dgg^{-1})_2(dgg^{-1})_2(gt_ag^{-1})_0)$ with two gamma-matrices. We get:

$$- \Gamma^m \Gamma^n \text{Str}([t_m^2, t_n^2](gt_ag^{-1})_0) \quad (352)$$

This expression can be simplified in the following way. Notice that we are considering B with only nonzero components in $so(6) \wedge so(6) \subset \mathfrak{g} \wedge \mathfrak{g}$. This means that the index a in (352) only runs in $so(6)_2 \subset \mathfrak{g}_2$. Besides that, we turned off all the thetas, so $g \in SO(2, 4) \times SO(6)$. This means that $(gt_ag^{-1})_0 \in so(5) \subset so(6) \subset so(2, 4) \oplus so(6)$. In the spinor representation, the action of t_m^2 on \mathfrak{g}_{odd} is:

$$[t_m^2, \psi^\alpha t_\alpha^3] = \frac{1}{2}(\widehat{F}_+ \Gamma_m \psi)^{\hat{\beta}} t_{\hat{\beta}}^1 \\ [t_m^2, \psi^{\hat{\alpha}} t_{\hat{\alpha}}^1] = \frac{1}{2}(\widehat{F}_+ \Gamma_m \psi)^\beta t_\beta^3 \quad (353)$$

where $\widehat{F}_+ = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 + \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9$ is the bispinor associated to the RR field strength. This means that:

$$[t_m^2, t_n^2] = \overline{\Gamma}_{[m} \Gamma_{n]} \quad (354)$$

where

$$\bar{\Gamma}_m = \begin{cases} \Gamma_m & \text{for } m \in \{0, \dots, 4\} \\ -\Gamma_m & \text{for } m \in \{5, \dots, 9\} \end{cases} \quad (355)$$

Therefore, when $(gt_ag^{-1}) \in so(6)$, we get

$$\iota(\Gamma)\iota(\Gamma)\text{Str}((dgg^{-1})_2(dgg^{-1})_2(gt_ag^{-1})_0) = -\Gamma^m\Gamma^n\text{Str}([t_m^2, t_n^2](gt_ag^{-1})_0) = -2(gt_ag^{-1})_0 \quad (356)$$

Note that our supertrace, which we denote Str , is in the fundamental representation of $su(2, 2|4)$; therefore the trace in (356) should be understood as the trace in the spinor representation of $so(6)$, which is the fundamental of $su(4)$. Therefore $\text{tr } 1 = 4$. Both t_m and $[t_m, t_n]$ are conjugate to $\text{diag}(1/2, 1/2, -1/2, -1/2)$. Therefore $-\text{tr}(t_m t_n) = \delta_{mn}$ and $-\text{tr}([t_m, t_n][t_k, t_l]) = \delta_{mk}\delta_{nl} - \delta_{ml}\delta_{nk}$. In particular, $\Gamma^m\Gamma^n\text{Str}([t_m^2, t_n^2][t_k^2, t_l^2]) = [\Gamma_k, \Gamma_l] = 2[t_k^2, t_l^2]$.

Similarly, the contraction of H with two gamma-matrices gives:

$$\begin{aligned} \iota(\Gamma)\iota(\Gamma)H &= 8 \times 2 (gt_ag^{-1})_0 B^{ab} \text{Str}((dgg^{-1})_2(gt_bg^{-1})_2) + \\ &\quad + 8 \times 2 B^{ab}[(gt_ag^{-1})_0, (dgg^{-1})_2], (gt_bg^{-1})_2] = \\ &= 8 \times 2 (gt_ag^{-1})_0 B^{ab} \text{Str}((dgg^{-1})_2(gt_bg^{-1})_2) - \\ &\quad - 8 D_0(B^{ab}[(gt_ag^{-1})_2, (gt_bg^{-1})_2]) = \\ &= 8 (gt_ag^{-1})_0 B^{ab} * j_b - \\ &\quad - 8 D_0(B^{ab}[(gt_ag^{-1})_2, (gt_bg^{-1})_2]) \end{aligned} \quad (357)$$

Now we see that:

- the first term in (357) reproduces the bosonic part of the coupling (347)
- the second term is missing from (347).

But in fact, that second term is a total derivative: $D_0(-8 B^{ab}[(gt_ag^{-1})_2, (gt_bg^{-1})_2])$ and can be absorbed into the redefinition of λ and w . More precisely, the pure spinor kinetic term:

$$\begin{aligned} \mathcal{L}_{w\lambda} &= \text{Str}(w_{1+}(D_{0-}\lambda_3 + 4j_{a-}B^{ab}[(gt_bg^{-1})_0, \lambda_3])) + \\ &\quad + \text{Str}(w_{3-}(D_{0+}\lambda_1 + 4j_{a+}B^{ab}[(gt_bg^{-1})_0, \lambda_1])) \end{aligned} \quad (358)$$

can be rewritten as follows:

$$\begin{aligned} \mathcal{L}_{w\lambda} &= \text{Str}\left(w_{1+}\left(D_{0-} + \frac{1}{2}(\iota(\Gamma)\iota(\Gamma)H_-) + 4D_0(B^{ab}[(gt_ag^{-1})_2, (gt_bg^{-1})_2])\right)\lambda_3\right) + \\ &\quad + \text{Str}\left(w_{3-}\left(D_{0-} - \frac{1}{2}(\iota(\Gamma)\iota(\Gamma)H_+) - 4D_0(B^{ab}[(gt_ag^{-1})_2, (gt_bg^{-1})_2])\right)\lambda_1\right) \end{aligned} \quad (359)$$

Therefore the term with the total derivative $D_0(B^{ab}[(gt_ag^{-1})_2, (gt_bg^{-1})_2])$ can be removed by the following infinitesimal field redefinition:

$$\begin{aligned}\delta\lambda_3 &= [\Phi, \lambda_3] & , & \quad \delta w_{1+} = [\Phi, w_{1+}] \\ \delta\tilde{\lambda}_1 &= -[\Phi, \tilde{\lambda}_1] & , & \quad \delta\tilde{w}_{3-} = -[\Phi, \tilde{w}_{3-}]\end{aligned}\tag{360}$$

where the parameter Φ is:

$$\Phi = 4B^{ab}[(gt_ag^{-1})_2, (gt_bg^{-1})_2]\tag{361}$$

11 Relation to the description by Alday–Arutyunov–Frolov

Generally speaking, let us consider a theory with some action S_0 and symmetry algebra \mathfrak{g} . Suppose that we act by the symmetry transformations with some constant parameters ϵ^a . Then the action is invariant:

$$S[\phi + \epsilon^a t_a \phi] = S[\phi]\tag{362}$$

Now suppose that ϵ^a is not a constant but depends on τ^+ and τ^- . Then the variation of the action is:

$$S[\phi + \epsilon^a t_a \phi] = S[\phi] + \int \int d\tau^+ d\tau^- (j_{a+} \partial_- \epsilon^a + j_{a-} \partial_+ \epsilon^a)\tag{363}$$

(This formula holds off-shell.) In particular, let us consider ϵ^a satisfying these equations:

$$\partial_+ \epsilon^a = \frac{1}{2} B^{ab} j_{b+}\tag{364}$$

$$\partial_- \epsilon^a = -\frac{1}{2} B^{ab} j_{b-}\tag{365}$$

Then Eq. (363) tells us that the change of variables $\phi \mapsto \tilde{\phi}$:

$$\phi = \tilde{\phi} + \epsilon^a t_a \tilde{\phi} + \dots\tag{366}$$

transforms S into the deformed action:

$$S[\phi] = S[\tilde{\phi}] + \int \int d\tau^+ d\tau^- B^{ab} j_{a+} j_{b-}\tag{367}$$

When we solve for ϵ satisfying (364) and (365) the resulting ϵ will not be periodic in σ . Indeed, the deviation from the periodicity will accumulate, and is given by the integral:

$$\int d\sigma B^{ab} j_{b\tau} = \epsilon^a(\tau, \sigma + 2\pi) - \epsilon^a(\tau, \sigma)\tag{368}$$

This results to the twisted boundary conditions of [27].

Note that we define ϵ by Eqs. (364) and (365), but these equations are compatible only on-shell. Indeed, the compatibility condition of (364) and (365) is:

$$\partial_- j_{a+} + \partial_+ j_{a-} = 0 \quad (369)$$

which is precisely the currents conservation, and only holds on-shell. Therefore, this is only limited to calculating the value of the action on the classical solution.

Our approach, on the other hand, is not limited to the classical configurations.

12 General relation between NSNS and RR fields in the β -deformed solution

In this section we will discuss the relation between the NSNS and the RR fields of the Maldacena-Lunin solution. It will turn out that the RR 3-form dC_2 is in fact Hodge dual to the B^{NS} :

$$B_{ij}^{NS} = c \epsilon_{ij}{}^{klm} \partial_k C_{lm} \quad (370)$$

where c is some coefficient, which is constant at the linearized level, but becomes a function in the full nonlinear solution.

12.1 At the linearized level

We will start by discussing the relation (370) at the first order in the deformation parameter. Notice that Maldacena and Lunin identify φ_1 and φ_2 as cyclic coordinates corresponding to their two $U(1)$ symmetries; those two $U(1)$ act as $\frac{\partial}{\partial \varphi_1}$ and $\frac{\partial}{\partial \varphi_2}$. On the other hand Eq. (3.2) of their paper implies that in terms of ϕ_1 and ϕ_2 they act as:

$$\frac{\partial}{\partial \varphi_1} = \frac{\partial}{\partial \phi_2} - \frac{\partial}{\partial \phi_3}, \quad \frac{\partial}{\partial \varphi_2} = -\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} \quad (371)$$

For any vector field v^μ , we can contract it with the metric and get a one form $g(v)$, defined as: $g(v)_\mu = g_{\mu\nu} v^\nu$. Then Eq. (3.11) from Maldacena-Lunin implies:

$$g\left(\frac{\partial}{\partial \varphi_1}\right) = \mu_2^2 d\phi_2 - \mu_3^2 d\phi_3 \quad (372)$$

$$g\left(\frac{\partial}{\partial \varphi_2}\right) = -\mu_1^2 d\phi_1 + \mu_2^2 d\phi_2 \quad (373)$$

Now we see:

$$B^{NS} = \hat{\gamma} R^2 g\left(\frac{\partial}{\partial \varphi_1}\right) \wedge g\left(\frac{\partial}{\partial \varphi_2}\right) \quad (374)$$

Then we can use that for any two vector fields v and u :

$$*(g(v) \wedge g(u)) = \iota_v \iota_u (\text{volume-form}) \quad (375)$$

The volume form of S^5 in terms of $(\alpha, \theta, \psi, \varphi_1, \varphi_2)$ is:

$$s_\alpha^3 c_\alpha c_\theta s_\theta \, d\alpha \wedge d\theta \wedge d\psi \wedge d\phi_1 \wedge d\phi_2 \quad (376)$$

This equation with our Eq. (374) and equation for C_2 on p.9 of Maldacena and Lunin imply our Eq. (370).

12.2 Exact relation for the full solution

The deformed metric is:

$$\begin{aligned} ds_\gamma^2 = & d\alpha^2 + s_\alpha^2 d\theta^2 + G(1 + 9\hat{\gamma}^2 s_\alpha^4 c_\alpha^2 s_\theta^2 c_\theta^2) d\psi^2 + \\ & + G s_\alpha^2 d\varphi_1^2 + G(s_\alpha^2 c_\theta^2 + c_\alpha^2) d\varphi_2^2 + 2G s_\alpha^2 (c_\theta^2 - s_\theta^2) d\psi d\varphi_1 + \\ & + 2G(s_\alpha^2 c_\theta^2 - c_\alpha^2) d\psi d\varphi_2 + 2G s_\alpha^2 c_\theta^2 d\varphi_1 d\varphi_2 \end{aligned} \quad (377)$$

where

$$G = \frac{1}{1 + \hat{\gamma}^2 (\mu_1^2 \mu_2^2 + \mu_1^2 \mu_3^2 + \mu_2^2 \mu_3^2)} = \frac{1}{1 + \hat{\gamma}^2 s_\alpha^2 (c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2)}. \quad (378)$$

We may write it as:

$$\begin{aligned} ds_\gamma^2 = & d\alpha^2 + (\mu_2^2 + \mu_3^2) d\theta^2 + G(1 + 9\hat{\gamma}^2 \mu_1^2 \mu_2^2 \mu_3^2) d\psi^2 + G(\mu_2^2 + \mu_3^2) d\varphi_1^2 + \\ & + G(\mu_1^2 + \mu_2^2) d\varphi_2^2 + 2G(\mu_2^2 - \mu_3^2) d\psi d\varphi_1 + \\ & + 2G(\mu_2^2 - \mu_1^2) d\psi d\varphi_2 + 2G\mu_2^2 d\varphi_1 d\varphi_2. \end{aligned} \quad (379)$$

The determinant of the metric above is

$$g_\gamma = 9\mu_1^2 \mu_2^2 \mu_3^2 G^2 (\mu_2^2 + \mu_3^2), \quad (380)$$

and the non-zero components of the inverse matrix are:

$$\begin{aligned} g_\gamma^{\alpha\alpha} &= 1 \\ g_\gamma^{\theta\theta} &= \frac{1}{\mu_2^2 + \mu_3^2} \\ g_\gamma^{\psi\psi} &= \frac{1}{9\mu_1^2 \mu_2^2 \mu_3^2} (\mu_1^2 \mu_2^2 + \mu_1^2 \mu_3^2 + \mu_2^2 \mu_3^2) \\ g_\gamma^{\psi\varphi_1} &= \frac{1}{9\mu_1^2 \mu_2^2 \mu_3^2} (\mu_1^2 \mu_3^2 + \mu_2^2 \mu_3^2 - 2\mu_1^2 \mu_2^2) \\ g_\gamma^{\psi\varphi_2} &= \frac{1}{9\mu_1^2 \mu_2^2 \mu_3^2} (\mu_1^2 \mu_2^2 + \mu_1^2 \mu_3^2 - 2\mu_2^2 \mu_3^2) \\ g_\gamma^{\varphi_1\varphi_1} &= (\mu_1^2 + \mu_2^2) \hat{\gamma}^2 + \frac{\mu_1^2 + \mu_2^2 - (\mu_1^2 - \mu_2^2)^2}{9\mu_1^2 \mu_2^2 \mu_3^2} \\ g_\gamma^{\varphi_2\varphi_2} &= (\mu_2^2 + \mu_3^2) \hat{\gamma}^2 + \frac{\mu_2^2 + \mu_3^2 - (\mu_2^2 - \mu_3^2)^2}{9\mu_1^2 \mu_2^2 \mu_3^2} \\ g_\gamma^{\varphi_1\varphi_2} &= -\mu_2^2 \hat{\gamma}^2 + \frac{\mu_2^4 + \mu_1^2 \mu_2^2 - \mu_2^2 \mu_3^2 - \mu_2^2}{9\mu_1^2 \mu_2^2 \mu_3^2} \end{aligned}$$

The RR field strength $F_3 = dC_2$ is given by:

$$F_3 = \frac{1}{3!} F_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \quad (381)$$

where the only non-zero component is $F_{\alpha\theta\psi} = -12\gamma\pi N s_\alpha^2 s_{2\alpha} s_{2\theta}$. Its Hodge dual on the deformed sphere S_γ^5 is:

$$\begin{aligned} {}^*F_3 &= \sqrt{g_\gamma} g_\gamma^{\alpha\alpha} g_\gamma^{\theta\theta} F_{\alpha\theta\psi} \left(g_\gamma^{\psi\psi} d\varphi_1 \wedge d\varphi_2 - g_\gamma^{\varphi_1\psi} d\psi \wedge d\varphi_2 - g_\gamma^{\varphi_2\psi} d\varphi_1 \wedge d\psi \right) \\ &= -16\gamma\pi N G \left((\mu_1^2 \mu_2^2 + \mu_1^2 \mu_3^2 + \mu_2^2 \mu_3^2) d\varphi_1 \wedge d\varphi_2 + \right. \\ &\quad \left. + (\mu_1^2 \mu_2^2 - 2\mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2) d\psi \wedge d\varphi_1 + \right. \\ &\quad \left. + (2\mu_1^2 \mu_2^2 - \mu_2^2 \mu_3^2 - \mu_3^2 \mu_1^2) d\psi \wedge d\varphi_2 \right). \quad (382) \end{aligned}$$

From $B_2 = c {}^*F_3$ and

$$\begin{aligned} B_2 &= \hat{\gamma} R^2 G \left((\mu_1^2 \mu_2^2 + \mu_1^2 \mu_3^2 + \mu_2^2 \mu_3^2) d\varphi_1 \wedge d\varphi_2 + \right. \\ &\quad \left. + (\mu_1^2 \mu_2^2 - 2\mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2) d\psi \wedge d\varphi_1 + \right. \\ &\quad \left. + (2\mu_1^2 \mu_2^2 - \mu_2^2 \mu_3^2 - \mu_3^2 \mu_1^2) d\psi \wedge d\varphi_2 \right), \quad (383) \end{aligned}$$

we get

$$c = -\frac{R^4}{16\pi N} \quad (384)$$

A Antisymmetric tensor product of two adjoint representations of $su(4)$

A.1 As a representation of $su(4)$

Consider the antisymmetric tensor product of two adjoint representations of $su(4)$. The adjoint of $su(4)$ is the traceless part of the tensor product of the fundamental and the complex conjugate of the fundamental. The elements of the adjoint of $su(4)$ can be written as u_j^i where the upper index i is the fundamental of $su(4)$, and the lower index is the complex conjugate of the fundamental. The anti-hermiticity condition is:

$$(u_j^i)^* = -u_i^j \quad (385)$$

and the traceless condition is $u_i^i = 0$.

The antisymmetric tensor product of two adjoint representation is a subspace in the space of tensors b_{jl}^{ik} such that:

$$b_{jl}^{ik} = -b_{lj}^{ki} \quad (386)$$

satisfying the trace condition:

$$b_{il}^{ik} = 0 \quad (387)$$

and the reality condition:

$$(b_{jl}^{ik})^* = b_{ik}^{jl} \quad (388)$$

The adjoint representation is a subspace:

$$\text{ad} \subset \text{ad} \wedge \text{ad} \quad (389)$$

consisting of the tensors of the following form:

$$b_{jl}^{ik} = \delta_l^i b_j^k - \delta_j^k b_l^i \quad (390)$$

where b_j^i can be chosen to satisfy: $b_i^i = 0$. To project on the orthogonal subspace we impose the additional trace condition on b :

$$b_{pj}^{ip} = 0 \quad (391)$$

Every tensor satisfying (386) can be represented in a unique way as a sum of a tensor antisymmetric in lower indices, and symmetric in upper indices, and a tensor antisymmetric in upper indices, and symmetric in lower indices:

$$b_{jl}^{ik} = x_{jl}^{ik} + y_{jl}^{ik} \quad (392)$$

$$x_{jl}^{ik} = x_{jl}^{ki} = -x_{lj}^{ik} := b_{jl}^{(ik)} \quad (393)$$

$$y_{jl}^{ik} = -y_{jl}^{ki} = y_{lj}^{ik} := b_{jl}^{[ik]} \quad (394)$$

To project on the subspace orthogonal to (389) we impose the constraint:

$$x_{pj}^{pi} = y_{pj}^{pi} = 0 \quad (395)$$

This is equivalent to the additional trace condition (391) on b . Moreover, the reality condition (388) relates y to x :

$$y_{jl}^{ik} = (x_{ik}^{jl})^* \quad (396)$$

Lemma: There is a one-to-one correspondence between complex tensors x_{jl}^{ik} symmetric in the upper indices ik and antisymmetric in the lower indices jl , and tensors b_{jl}^{ik} satisfying the antisymmetry condition (386) and the *reality condition* (388).

Notice that there is no reality condition on x_{jl}^{ik} . This means that the irreducible component in the antisymmetric product of two adjoints can be described in terms of the *complex* tensor x_{jl}^{ik} satisfying the following symmetry and tracelessness conditions:

$$x_{jl}^{ik} = x_{jl}^{ki} = -x_{lj}^{ik} \quad (397)$$

$$x_{pj}^{pi} = 0 \quad (398)$$

There are 45 linearly independent (over \mathbf{C}) tensors x_{jl}^{ik} satisfying (397) and (398), therefore this representation is called $45_{\mathbf{C}}$.

Conclusion: The antisymmetric tensor product of two adjoint representations contains a 90-dimensional irreducible component, which is actually defined over \mathbf{C} , and is denoted $45_{\mathbf{C}}$.

A.2 As a representation of $u(3) \subset su(4)$

Let us agree that the superindex of the representation symbol indicates a representation of which algebra we are considering. So, the $45_{\mathbf{C}}$ of $su(4)$ will be denoted:

$$45_{\mathbf{C}}^{su(4)}$$

This representation splits into several irreducible representations of $u(3) \subset su(4)$. For example, there is a representation $6_{\mathbf{C}}^{su(3)}$ which is realized on the symmetric complex tensors u^{ijk} where the indices i, j, k enumerate the fundamental representation of $u(3)$. We observe:

$$6_{\mathbf{C}}^{u(3)} \subset 45_{\mathbf{C}}^{su(4)} \quad (399)$$

In terms of x_{jl}^{ik} this is:

$$x_{jl}^{ik} = u^{ikp} \epsilon_{pjl} \quad (400)$$

B BRST operator in the near flat space expansion

We will use the “most symmetric” gauge:

$$g = e^X \quad (401)$$

where $X = x_2 + \theta_3 + \theta_1$. The BRST operator acts on g in the following way:

$$\epsilon Q g = (\epsilon \lambda_3 + \epsilon \lambda_1) g = (e^{X+\epsilon Q X} - e^X) + \omega_0(\epsilon) e^X \quad (402)$$

— this equation is the definition of $\epsilon Q X$ and $\omega_0(\epsilon)$.

$$\epsilon \lambda - \omega_0(\epsilon) = \frac{e^{\text{ad}(X)} - 1}{\text{ad}(X)} \epsilon Q X \quad (403)$$

This gives us the following recursive formulas for QX and ω_0 :

$$\epsilon QX = \epsilon\lambda - \frac{1}{2}[X, \epsilon QX]_\perp - \frac{1}{6}[X, [X, \epsilon QX]]_\perp - \frac{1}{24}[X, [X, [X, \epsilon QX]]]_\perp - \dots \quad (404)$$

$$\omega_0(\epsilon) = -\frac{1}{2}[X, \epsilon QX]_0 - \frac{1}{6}[X, [X, \epsilon QX]]_0 - \frac{1}{24}[X, [X, [X, \epsilon QX]]]_0 - \dots \quad (405)$$

This gives us:

$$\begin{aligned} \epsilon QX &= \epsilon\lambda - \frac{1}{2}[X, \epsilon\lambda]_\perp + \\ &+ \frac{1}{12}[X, [X, \epsilon\lambda]_\perp]_\perp - \frac{1}{6}[X, [X, \epsilon\lambda]_0]_\perp + \\ &+ \frac{1}{24}\left([X, [X, [X, \epsilon\lambda]_0]_\perp]_\perp + [X, [X, [X, \epsilon\lambda]_\perp]_0]_\perp\right) + \dots \end{aligned} \quad (406)$$

We observe that to preserve the gauge (401) we need to combine the action of Q_{BRST} with the gauge transformation with the parameter $-\omega_0$:

$$\delta_\epsilon g = -\omega_0(\epsilon)g \quad (407)$$

We will include this “compensating” gauge transformation into the definition of Q_{BRST} . Then, in particular, our “combined” BRST transformation *acts on the pure spinor*:

$$\epsilon Q\epsilon'\lambda = [\epsilon'\lambda, \omega_0(\epsilon)] \quad (408)$$

C Proof of a technical Lemma used in Section 7.4.3

Here we will prove that any tensor of the form $A^{p[a} f_p^{bc]}$ should necessarily have some fermionic indices. It is useful to write this in the matrix notations. Let i, j, \dots denote the indices of the upper left square (the fundamental representation of $su(2, 2)$), and α, β, \dots denote the indices of the lower right square (the fundamental representation of $su(4)$). Then the generators E_j^i and E_β^α are bosonic, and the generators E_α^i and E_j^α are fermionic. We want to see if there exists A^{pa} such that $Q(A^{pa}\bar{c}_p c_a) = A^{pa} f_p^{bc} c_a c_b c_c$ contains only c_j^i and c_β^α but not c_α^i and c_i^α . In matrix notations:

$$A^{pa}\bar{c}_p c_a = A^{ik}\bar{c}_i^j c_k^l + A_{\alpha k}^{i\beta}\bar{c}_i^\alpha c_\beta^k + \dots \quad (409)$$

Then:

$$Q(A^{pa}\bar{c}_p c_a)|_{\text{mixed}} = A_{ik}^{jl} c_{\alpha}^i c_j^{\alpha} c_l^k + A_{\alpha k}^{\beta l} c_{\overline{m}}^\alpha c_\beta^{\overline{m}} c_l^k + \quad (410)$$

$$+ A_{i\alpha}^{j\beta} c_{\overline{n}}^i c_j^{\overline{n}} c_\beta^\alpha + A_{\gamma\alpha}^{\delta\beta} c_{\overline{m}}^\gamma c_\delta^{\overline{m}} c_\beta^\alpha + \quad (411)$$

$$+ A_{ij}^{\alpha\beta} (c_{\overline{n}}^i c_\alpha^{\overline{n}} c_\beta^j + c_{\overline{m}}^i c_\alpha^{\overline{m}} c_\beta^j) + A_{\alpha\beta}^{ij} (c_{\overline{m}}^\alpha c_i^{\overline{m}} c_j^\beta + c_{\overline{n}}^\alpha c_i^{\overline{n}} c_j^\beta) + \quad (412)$$

$$+ A_{\alpha j}^{i\beta} (c_{\overline{m}}^\alpha c_i^{\overline{m}} c_\beta^j + c_{\overline{n}}^\alpha c_i^{\overline{n}} c_\beta^j) + A_{i\beta}^{\alpha j} (c_{\overline{m}}^i c_\alpha^{\overline{m}} c_j^\beta + c_{\overline{n}}^i c_\alpha^{\overline{n}} c_j^\beta) \quad (413)$$

where we put boxes around the summation indices, just to make them clearly visible. We require that these mixed terms all cancel. Because of the different tensor structure of different terms, this actually implies separate cancellations:

$$0 = A_{ik}^{jl} c_{\boxed{i}}^i c_{\boxed{j}}^j c_l^k + A_{\alpha k}^{\beta l} c_{\boxed{\alpha}}^{\alpha} c_{\boxed{\beta}}^{\beta} c_l^k + A_{\alpha j}^{i \beta} c_{\boxed{\alpha}}^{\alpha} c_{\boxed{i}}^i c_{\beta}^j + A_{i \beta}^{\alpha j} c_{\boxed{\alpha}}^{\alpha} c_{\boxed{\beta}}^{\beta} c_j^i \quad (414)$$

$$0 = A_{i \alpha}^{j \beta} c_{\boxed{i}}^i c_{\boxed{\alpha}}^{\alpha} c_{\beta}^j + A_{\gamma \alpha}^{\delta \beta} c_{\boxed{\gamma}}^{\gamma} c_{\boxed{\alpha}}^{\alpha} c_{\beta}^{\delta} + A_{\alpha j}^{i \beta} c_{\boxed{\alpha}}^{\alpha} c_{\boxed{i}}^i c_{\beta}^j + A_{i \beta}^{\alpha j} c_{\boxed{\alpha}}^{\alpha} c_{\boxed{\beta}}^{\beta} c_j^i \quad (415)$$

$$0 = A_{ij}^{\alpha \beta} c_{\boxed{i}}^i c_{\boxed{j}}^j c_{\alpha}^{\alpha} c_{\beta}^{\beta} \quad (416)$$

$$0 = A_{ij}^{\alpha \beta} c_{\boxed{i}}^i c_{\boxed{\alpha}}^{\alpha} c_{\beta}^j \quad (417)$$

$$0 = A_{\alpha \beta}^{i j} c_{\boxed{\alpha}}^{\alpha} c_{\boxed{\beta}}^{\beta} c_i^i c_j^j \quad (418)$$

$$0 = A_{\alpha \beta}^{i j} c_{\boxed{\alpha}}^{\alpha} c_{\boxed{i}}^i c_j^{\beta} \quad (419)$$

Eqs. (416) — (419) imply that:

$$A_{ij}^{\alpha \beta} = A_{i j}^{\alpha \beta} = A_{\alpha \beta}^{i j} = A_{\alpha \beta}^{i j} = 0 \quad (420)$$

For example, $A_{ij}^{\alpha \beta} = 0$ can be proven as follows. We can think $A_{ij}^{\alpha \beta} = 0$ as a tensor in $L_{dn} \otimes L_{dn} \otimes L'_{up} \otimes L'_{up}$. Suppose that we can find a nonzero A such that (416) is satisfied. The space of such A is a subspace in $L_{dn} \otimes L_{dn} \otimes L'_{up} \otimes L'_{up}$ closed under the action of $su(2, 2) \oplus su(4)$. But in fact there are only four such invariant subspaces:

$$\Lambda^2 L_{dn} \otimes \Lambda^2 L'_{up}, \quad S^2 L_{dn} \otimes \Lambda^2 L'_{up}, \quad \Lambda^2 L_{dn} \otimes S^2 L'_{up}, \quad S^2 L_{dn} \otimes S^2 L'_{up} \quad (421)$$

and we can show by a direct examination that neither of these solves (416). For example the first one corresponds to considering the antisymmetrization of ij and $\alpha\beta$:

$$c_{\boxed{i}}^{[i]} c_{\boxed{\alpha}}^{[\alpha]} c_{\boxed{j}}^{j]} c_{\boxed{\beta}}^{\beta]}$$

which does not give zero. The other three possibilities (*e.g.* symmetrizing ij and antisymmetrizing $\alpha\beta$ also do not give zero); therefore the only solution to (416) is $A_{ij}^{\alpha \beta} = 0$. The other three identities in (420) can be proven in a similar way.

The analysis of (414) and (415) is slightly more complicated, because besides symmetrization/antisymmetrization one can also use the Kronecker δ_j^i and δ_{β}^{α} . Just symmetrization/antisymmetrization does not work, but there are nonzero solutions for A involving the Kronecker delta. Indeed, let us write the most general ansatz with the Kronecker delta:

$$\begin{aligned} A^{pa} \bar{c}_p c_a &= \text{tr}(\bar{c}_u^u x_u^u c_u^u) + \text{tr}(c_u^u \tilde{x}_u^u \bar{c}_u^u) + \\ &+ \text{tr}(\bar{c}_d^d y_d^d c_u^d) + \text{tr}(c_d^d \tilde{y}_d^d \bar{c}_u^d) + \\ &+ \text{tr}(\bar{c}_u^d z_u^u c_d^u) + \text{tr}(c_u^d \tilde{z}_u^u \bar{c}_d^u) + \\ &+ \text{tr}(\bar{c}_d^d w_d^d c_d^d) + \text{tr}(c_d^d \tilde{w}_d^d \bar{c}_d^d) \end{aligned} \quad (422)$$

where we introduced the 4×4 -matrices of ghosts:

$$c_u^u = \sum_{i,j=1}^4 c_j^i E_i^j, \quad c_d^u = \sum_{i,\alpha=1}^4 c_\alpha^i E_i^\alpha, \quad \text{etc.} \quad (423)$$

and $x, y, z, w, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ are some coefficients. To exemplify our notations, we write explicitly some terms:

$$\begin{aligned} \text{tr}(\bar{c}_u^u x_u^u c_u^u) &= \bar{c}_j^i x_k^j c_i^k \\ \text{tr}(\bar{c}_u^d z_u^u c_d^u) &= \bar{c}_i^\alpha z_j^i c_\alpha^j \end{aligned} \quad (424)$$

Let us impose the condition that $QA^{pa}\bar{c}_p c_a$ does not contain any fermionic ghosts. This leads to the following equations:

$$\begin{aligned} \text{from } c_u^u c_d^u c_u^d : & \quad x_u^u + \tilde{z}_u^u = 0 \\ \text{from } c_d^u c_u^d c_u^u : & \quad \tilde{x}_u^u - z_u^u = 0 \\ \text{from } c_u^d c_u^u c_d^u : & \quad y_d^d - \tilde{y}_d^d = 0 \\ \text{from } c_u^d c_d^u c_d^d : & \quad -y_d^d - \tilde{w}_d^d = 0 \\ \text{from } c_d^d c_u^d c_d^u : & \quad \tilde{y}_d^d + w_d^d = 0 \\ \text{from } c_d^u c_d^d c_u^d : & \quad z_u^u - \tilde{z}_u^u = 0 \end{aligned} \quad (425)$$

This means that (422) collapses to this:

$$\text{tr} \left(x_u^u c_u^u \bar{c}_u^u + x_u^u \bar{c}_u^u c_u^u + x_u^u \bar{c}_d^u c_u^d + x_u^u c_d^u \bar{c}_u^d \right) + \quad (426)$$

$$+ \text{tr} \left(y_d^d c_d^d \bar{c}_d^d + y_d^d \bar{c}_d^d c_d^d + y_d^d \bar{c}_u^d c_d^u + y_d^d c_u^d \bar{c}_d^u \right) \quad (427)$$

and one can see that Q of this is actually zero. (Not only Q of (426) and (427) does not contain c_d^u and c_u^d , but actually it is just 0.) This proves the **Lemma**.

Acknowledgments

We want to thank Y. Aisaka, N.J. Berkovits, A. Losev, L. Mazzucato and A. Tseytlin for many useful discussions. O.A.B. would like to thank FAPESP grant 09/08893-9 for financial support, as well as the Aspen Center of Physics for hospitality during the workshop ‘‘Unity of String Theory’’. The research of L.I.B. is supported by CNPq Grant No. 141546/2006-9. The research of A.M. was supported by the Sherman Fairchild Fellowship and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan, and in part by the RFBR Grant No. 10-02-01315. The work of V.O.R. is supported by CNPq grant No. 304495/2007-7, FAPESP grant No. 2008/05343-5 and PROSUL grant No. 490134/2006-8.

References

- [1] O. Aharony, B. Kol, and S. Yankielowicz, *On exactly marginal deformations of $N = 4$ SYM and type IIB supergravity on $AdS(5) \times S^{**}5$* , *JHEP* **06** (2002) 039, [[hep-th/0205090](#)].
- [2] N. Berkovits, *Simplifying and Extending the $AdS(5) \times S(5)$ Pure Spinor Formalism*, [arXiv:0812.5074](#).
- [3] A. Mikhailov, *Symmetries of massless vertex operators in $AdS(5) \times S(5)$* , [0903.5022](#).
- [4] N. Berkovits, *Perturbative Super-Yang-Mills from the Topological $AdS(5) \times S(5)$ Sigma Model*, *JHEP* **09** (2008) 088, [[0806.1960](#)].
- [5] P. A. Grassi and J. Kluson, *Pure spinor strings in TsT deformed background*, *JHEP* **03** (2007) 033, [[hep-th/0611151](#)].
- [6] S. A. Frolov, R. Roiban, and A. A. Tseytlin, *Gauge - string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045, [[hep-th/0503192](#)].
- [7] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069, [[hep-th/0503201](#)].
- [8] S. A. Frolov, R. Roiban, and A. A. Tseytlin, *Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B731** (2005) 1–44, [[hep-th/0507021](#)].
- [9] R. G. Leigh and M. J. Strassler, *Exactly marginal operators and duality in four-dimensional $N=1$ supersymmetric gauge theory*, *Nucl. Phys.* **B447** (1995) 95–136, [[hep-th/9503121](#)].
- [10] T. McLoughlin and I. Swanson, *Integrable twists in AdS/CFT* , *JHEP* **08** (2006) 084, [[hep-th/0605018](#)].
- [11] I. Swanson, *A review of integrable deformations in AdS/CFT* , *Mod. Phys. Lett.* **A22** (2007) 915–930, [[0705.2844](#)].
- [12] O. Lunin and J. M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033, [[hep-th/0502086](#)].
- [13] I. M. Gelfand and I. J. Dorfman, *Schouten bracket and Hamiltonian operators*, *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 3 71–74.
- [14] M. Cahen, S. Gutt, and J. Rawnsley, *Some remarks on the classification of Poisson Lie groups, Symplectic geometry and quantization (Sanda and Yokohama, 1993)*, *Contemp. Math.* , **179**, 1–16.

- [15] Y. Agaoka, *Left invariant Poisson structures on classical non-compact simple Lie groups*, *Israel J. Math.* **116** (2000) 189–222.
- [16] F. Elmetti, A. Mauri, S. Penati, and A. Santambrogio, *Conformal invariance of the planar beta-deformed $N = 4$ SYM theory requires beta real*, *JHEP* **01** (2007) 026, [[hep-th/0606125](#)].
- [17] O. Lunin, *Gravitational description of field theories*, *Nucl. Phys. Proc. Suppl.* **171** (2007) 99–118.
- [18] N. Berkovits, *BRST cohomology and nonlocal conserved charges*, *JHEP* **02** (2005) 060, [[hep-th/0409159](#)].
- [19] N. Berkovits, *Quantum consistency of the superstring in $AdS(5) \times S(5)$ background*, *JHEP* **03** (2005) 041, [[hep-th/0411170](#)].
- [20] A. Mikhailov and S. Schafer-Nameki, *Perturbative study of the transfer matrix on the string worldsheet in $AdS(5) \times S(5)$* , [arXiv:0706.1525](#) [[hep-th](#)].
- [21] N. Berkovits and C. Vafa, *Towards a Worldsheet Derivation of the Maldacena Conjecture*, *JHEP* **03** (2008) 031, [[0711.1799](#)].
- [22] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, *The Mass Spectrum of Chiral $N=2$ $D=10$ Supergravity on S^{*5}* , *Phys. Rev.* **D32** (1985) 389.
- [23] N. Berkovits and O. Chandia, *Superstring vertex operators in an $ads(5) \times s(5)$ background*, *Nucl. Phys.* **B596** (2001) 185–196, [[hep-th/0009168](#)].
- [24] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer (1987) (Springer series in soviet mathematics).
- [25] N. Berkovits and P. S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, *Nucl. Phys.* **B635** (2002) 75–105, [[hep-th/0112160](#)].
- [26] O. A. Bedoya and O. Chandia, *One-loop conformal invariance of the type II pure spinor superstring in a curved background*, *JHEP* **01** (2007) 042, [[hep-th/0609161](#)].
- [27] L. F. Alday, G. Arutyunov, and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, *JHEP* **06** (2006) 018, [[hep-th/0512253](#)].